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Spectral Theory for the Differential Equations of Simple Birth and Death Processes

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SPECTRAL THEORY FOR THE DIFFERENTIAL EQUATIONS OF SIMPLE BIRTH AND DEATH PROCESSES

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The enumerably infinite system of differential equations describing a temporally homogeneous birth and death process in a population is treated as the limiting case of one or the other of two finite systems of equations. Starting from the expansion of a finite matrix in terms of its associated idempotents, the solutions of the infinite system are displayed in spectral form which, in general, is written as a Stieltjes integral involving a spectral function. This method facilitates the investigation of asymptotic values and of the ergodic property of the system.

When the birth- and death-rates satisfy certain conditions of regularity, the spectrum is discrete and the solution can be written down more explicitly. Concrete examples are given, where the system has two distinct solutions for any set of initial conditions.

Finally, our method is applied to the known case of linear growth and to a problem in the theory of queues, confirming a result and a conjecture by D. G. Kendall.

INTRODUCTION

This paper is concerned with the temporally homogeneous birth and death process in a population whose state (size), at an instant t, may be given by any one of the integers $0, 1, 2, \ldots$. As regards the notation and the definition of the relevant concepts we follow Feller, who, in chapter 17 of his book (Feller 1950), has given a very lucid introduction to the theory.

If $p_j(t)$ ($j = 0, 1, 2, \ldots$) is the probability that, at time $t\,(\geq 0)$, the system is in the state j, then the fundamental differential equations of the process can be written in the form

$$\frac{d}{dt}p_0(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t), \tag{0·1}$$

$$\frac{d}{dt}p_j(t) = \lambda_{j-1}p_{j-1}(t) - (\lambda_j + \mu_j)\,p_j(t) + \mu_{j+1}p_{j+1}(t), \tag{0·2}$$

($j = 1, 2, \ldots$), where $\lambda_0, \lambda_1, \lambda_2, \ldots; \quad \mu_1, \mu_2, \mu_3, \ldots,$

are the birth- and death-rates corresponding to the various states. Throughout the paper we assume that the λ's and μ's are *strictly positive constants* possibly with the exception of λ_0, which may be zero or positive. Processes in which $\lambda_0 > 0$ permit of a revival through 'immigration' after the population has become temporarily extinct. Such processes differ in many respects from those in which $\lambda_0 = 0$, where the state 0 is an absorbing barrier.

It is convenient to define

$$\lambda_{-1} = \mu_0 = 0, \tag{0·3}$$

which makes it possible to regard (0·1) as a special case of (0·2).

The differential equations (0·1) and (0·2) may be written in matrix form thus:

$$\frac{d}{dt} p(t) = p(t) A, \tag{0·4}$$

where

$$p(t) = (p_0(t), p_1(t), p_2(t), \ldots)$$

is an infinite row vector and

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_n & -(\lambda_n+\mu_n) & \lambda_n \\ & & & & \cdots & \cdots \end{pmatrix} \tag{0·5}$$

is an infinite matrix. We note that all row sums of A are zero.

The solutions of (0·4) must satisfy the further conditions

$$0 \leqslant p_j(t) \leqslant 1 \quad (t \geqslant 0), \tag{0·6}$$

$$\sum_{j=0}^{\infty} p_j(t) \leqslant 1 \quad (t \geqslant 0). \tag{0·7}$$

It is known (see Feller 1950, p. 369) that the inequality sign in (0·7) cannot in general be dispensed with.

In order to adapt the solution to any given set of initial conditions

$$p_j(0) = \omega_j \quad (j = 0, 1, 2, \ldots), \tag{0·8}$$

where

$$0 \leqslant \omega_j \leqslant 1, \quad \sum_{j=0}^{\infty} \omega_j \leqslant 1,$$

we construct an infinite sequence of standard solutions $p_{ij}(t)$ $(i = 0, 1, 2, \ldots)$ with the initial conditions

$$p_{ij}(0) = \delta_{ij}.$$

Thus

$$\frac{d}{dt} p_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - (\lambda_j + \mu_j) p_{ij}(t) + \mu_{j+1} p_{i,j+1}(t), \tag{0·9}$$

$$0 \leqslant p_{ij}(t) \leqslant 1 \quad (t \geqslant 0), \tag{0·10}$$

and

$$\sum_{j=0}^{\infty} p_{ij}(t) \leqslant 1 \quad (t \geqslant 0). \tag{0·11}$$

The functions

$$p_j(t) = \sum_{i=0}^{\infty} \omega_i p_{ij}(t) \quad (j = 0, 1, 2, \ldots) \tag{0·12}$$

then constitute a solution satisfying the initial conditions (0·8). For it is clear that, for fixed j, the infinite series (0·12) converges uniformly for $t \geqslant 0$, as also does the series

$$\sum_i \omega_i \frac{d}{dt} p_{ij}(t), \tag{0·13}$$

because, by (0·9) and (0·10),

$$\left|\frac{d}{dt} p_{ij}(t)\right| \leqslant \lambda_{j-1} + \lambda_j + \mu_j + \mu_{j+1} = L_j$$

say, so that (0·13) is majorized by

$$L_j \sum_i \omega_i \leqslant L_j.$$

We therefore have

$$\frac{d}{dt} p_j(t) = \sum_{i=0}^{\infty} \omega_i \frac{d}{dt} p_{ij}(t),$$

from which the assertion follows after substituting for $\frac{d}{dt} p_{ij}(t)$.

Summarizing these preliminary remarks we can state our problem as follows: *given the matrix* A (0·5), *it is required to find a matrix*

$$P(t) = (p_{ij}(t))$$

whose elements are differentiable functions of t ($\geqslant 0$) *satisfying* (0·10) *and* (0·11), *such that*

$$\frac{d}{dt} P(t) = P(t) A \tag{0·14}$$

and

$$P(0) = I, \tag{0·15}$$

where I *is the infinite unit matrix.*

In the theory of Markov processes it is customary to impose the further condition

$$P(t+\tau) = P(t) P(\tau) \quad (t, \tau \geqslant 0), \tag{0·16}$$

which is known as the *Chapman-Kolmogorov equation.*

It appears that the problem has been attacked from three different directions by previous writers.

(i) Using powerful methods Feller (1940) has given an existence proof for a very wide class of Markov processes, which contains our problem as a special case. His procedure is one of successive approximations. We have shown elsewhere (Reuter & Ledermann 1953, hereafter quoted as 'D') that for systems with an enumerable number of states the existence of a solution can be established with fairly elementary means. These proofs do not lend themselves to any explicit construction of the solution.

(ii) If A were a finite matrix the solution of (0·14) could be written down at once in the form

$$P(t) = \exp(tA) = \sum_{r=0}^{\infty} \frac{1}{r!} (tA)^r. \tag{0·17}$$

But in the case of an infinite matrix the series does not always converge, especially when the elements are unbounded. However, Arley (1943) and Arley & Borchsenius (1945) have shown that the validity of (0·17) is ensured, at least for a sufficiently small range of t, provided the quantities λ_n and μ_n do not grow faster than n.

(iii) By using a probability-generating function

$$\phi(z,t) = \sum_{i=0}^{\infty} z^i p_i(t),$$

Kendall (1949) and others have obtained explicit formulae for the solution in the cases where λ_n and μ_n are linear functions of n, more precisely, where

$$\lambda_n = \lambda n + \kappa, \quad \mu_n = \mu n. \tag{0·18}$$

It seems difficult to apply this method when λ_n and μ_n are more complicated functions of n.

In the present paper, solutions of the problem will be obtained by a procedure whose main feature may be described as follows: the infinite matrix equation (0·14) is regarded as the limit of a sequence of finite matrix equations of increasing dimensions. The solutions of the finite equations tend to functions which are seen to be solutions of the infinite system. Sequences of this kind may be constructed in many ways, but we shall here be concerned with only two such sequences.

(i) *The nth (complete) section of A* is the $(n+1)$-rowed matrix

$$A^{(n)} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu_n & -(\lambda_n+\mu_n) \end{pmatrix},$$

derived from A by deleting all elements except those occupying the first $n+1$ rows and columns. The system of differential equations

$$\frac{d}{dt} X(t) = X(t)\, A^{(n)} \quad (X(0) = I^{(n)})$$

has the unique solution $X = F^{(n)}(t) = \exp(tA^{(n)})$.

In order to examine the limit, when $n \to \infty$, we find it convenient to write this matrix in its spectral form, thus

$$F^{(n)}(t) = \sum_{r=0}^{n} \exp(t\alpha_r^{(n)})\, D_r^{(n)}, \tag{0·19}$$

where the $\alpha_r^{(n)}$ $(r = 0, 1, \ldots, n)$ are the eigenvalues of $A^{(n)}$ and the $D_r^{(n)}$ are the corresponding orthogonal idempotent matrices. Then (in a sense to be made more precise later) we have

$$\lim_{n\to\infty} F^{(n)}(t) = F(t),$$

and the infinite matrix $F(t)$ satisfies (0·14) and all the additional conditions.

(ii) By *the nth modified section of A* we mean the $(n+1)$-rowed matrix

$$B^{(n)} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu_n & -\mu_n \end{pmatrix},$$

which differs from $A^{(n)}$ merely in the last element. The system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = Y(t)\, B^{(n)} \quad (Y(0) = I^{(n)})$$

has the solution

$$Y = G^{(n)}(t) = \exp(tB^{(n)}),$$

or, in spectral form,

$$G^{(n)}(t) = \sum_{r=0}^{n} \exp(t\beta_r^{(n)})\, E_r^{(n)}, \tag{0·20}$$

where the $\beta_r^{(n)}$ and $E_r^{(n)}$ $(r = 0, 1, \ldots, n)$ are the eigenvalues and associated idempotents of $B^{(n)}$. We cannot prove that $G^{(n)}(t)$ tends to a limit as $n \to \infty$, but using a general result of Helly (1921) we can show that (0·20) always has a convergent subsequence

$$\lim_{k\to\infty} G^{(n_k)}(t) = G(t),$$

where the infinite matrix $G(t)$ is again a solution of our problem. It may happen that $F(t) \neq G(t)$.

The formal apparatus dealing with finite sections of both kinds is assembled in chapter I.

The limit process is carried out in chapter II. The finite sums (0·19) and (0·20) are first transformed into Stieltjes integrals involving spectral step-functions which, possibly after the selection of a subsequence, tend to the spectral functions of the solution of the infinite system.

The method of spectral resolution facilitates the computation of the asymptotic values $F(\infty)$ and $G(\infty)$, which can be written down explicitly in many cases. This concludes the general theory.

Chapter III is devoted to a discussion of a particular class of birth and death processes which we term *analytical*. In such a process, the λ_n and μ_n are subject to certain regularity conditions which can be summarized by

$$\frac{\lambda_{n+1}}{\lambda_n} = 1 + \frac{a}{n} + O\left(\frac{1}{n^2}\right),$$

$$\frac{\lambda_n}{\mu_n} = c\left(1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right),$$

where the *parameters* a, b and c (>0) are real constants. The theory of analytical processes is developed independently of chapter II, and explicit formulae are obtained for the solutions $F(t)$ and $G(t)$. *It will be shown that for certain values of the parameters these two solutions are distinct.* This phenomenon of non-uniqueness is interesting and incidentally answers a question asked by Doob (1945).

The final chapter, IV, contains some examples. In particular, we obtain by our method the solution of a problem in the theory of queues.

Chapter I. Finite sections

1. *Summary of matrix formulae*

It is convenient to collect here some well-known results about finite matrices which are fundamental for our subsequent analysis.

Let C be an $n \times n$ matrix with *distinct* eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_n$. Then there exist n non-zero row-vectors $x_1, x_2, \ldots, x_n$ such that

$$x_r C = \gamma_r x_r \quad (r = 1, 2, \ldots, n), \tag{1·1}$$

and n non-zero column vectors $y'_1, y'_2, \ldots, y'_n$ such that

$$Cy'_r = \gamma_r y'_r \quad (r = 1, 2, \ldots, n). \tag{1·2}$$

Since eigenvectors that correspond to distinct eigenvalues are linearly independent, the matrices

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = (y'_1, \ldots, y'_n)$$

are non-singular. By (1·1) and (1·2)

$$x_r C y'_s = \gamma_r x_r y'_s = \gamma_s x_r y'_s,$$

so that

$$x_r y'_s = y_s x'_r = 0, \tag{1·3}$$

if $r \neq s$. Let

$$\theta_r = x_r y'_r = y_r x'_r \quad (r = 1, 2, \ldots, n) \tag{1·4}$$

and introduce the diagonal matrix

$$\Theta = \operatorname{diag}(\theta_1, \theta_2, \ldots, \theta_n);$$

then

$$XY = \Theta, \tag{1·5}$$

whence it follows that Θ is non-singular and therefore

$$\theta_r \neq 0 \quad (r = 1, 2, \ldots, n).$$

We can combine (1·1) and (1·2) into the single matrix equation

$$XCY = \Theta\Gamma,$$

where

$$\Gamma = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n).$$

Hence

$$C = X^{-1}\Theta\Gamma Y^{-1} = Y\Gamma\Theta^{-1}X,$$

that is,

$$C = \sum_{r=1}^{n} \gamma_r \theta_r^{-1} y'_r x_r.$$

This result may be written in the form

$$C = \sum_{r=1}^{n} \gamma_r J_r, \tag{1·6}$$

where

$$J_r = \theta_r^{-1} y'_r x_r \quad (r = 1, 2, \ldots, n). \tag{1·7}$$

The matrices (1·7) have the property that

$$J_r^2 = J_r, \quad J_r J_s = 0 \quad (r \neq s). \tag{1·8}$$

They are called the *orthogonal idempotents* of C.

We note that each of the idempotents is associated with one of the eigenvalues. The system of idempotents is *complete* in the sense that

$$J_1 + J_2 + \ldots + J_n = I. \tag{1·9}$$

For if

$$J = \sum_{s=1}^{n} J_s,$$

it is easily verified that $x_r J = x_r$ $(r = 1, 2, \ldots, n)$, i.e. that $XJ = X$, which implies (1·9) because $|X| \neq 0$.

Equation (1·6) will be referred to as the *spectral resolution* of C. *It applies to any matrix with distinct eigenvalues.* From (1·6) we can obtain the spectral resolution of every positive power and hence of every integral function of C. In particular,

$$\exp C = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} C^{\nu} = \sum_{r=1}^{n} (\exp \gamma_r) J_r. \tag{1·10}$$

2. *The spectrum of the nth section*

In order to construct a solution of the infinite matrix equation (0·14) we first solve the finite matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = X(t) A^{(n)}, \tag{1·11}$$

where the $(n+1)$-rowed matrix

$$A^{(n)} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu_n & -(\lambda_n+\mu_n) \end{pmatrix} \tag{1·12}$$

is called the *nth* (*complete*) *section of* A (its rows and columns are numbered from 0 to n), and quantities connected with the solution of (1·11) will generally bear a superscript or a suffix n. The $(n+1)$-rowed unit matrix will be denoted by $I^{(n)}$. We observe that all row-sums of $A^{(n)}$ are zero, except the last, which is equal to $-\lambda_n$.

The solution of (1·11) with the appropriate initial conditions $X(0) = I^{(n)}$ can be written down in the form

$$X = \exp(tA^{(n)}) = F^{(n)}(t) = (f_{ij}^{(n)}(t)). \tag{1·13}$$

We shall now show that $f_{ij}^{(n)}(t) \geq 0$ when $t \geq 0$. In fact, let

$$q = \max(\lambda_0, \lambda_1+\mu_1, \ldots, \lambda_n+\mu_n).$$

Then

$$M = tqI^{(n)} + tA^{(n)}$$

is a matrix all of whose elements are non-negative. From the definition as a power series it is clear that the elements of $\exp M$ are likewise non-negative. Hence so also are the elements of

$$\exp(tA^{(n)}) = \exp(-tqI^{(n)}) \exp M,$$

since

$$\exp(-tqI^{(n)}) = \exp(-tq) I^{(n)}.$$

Thus we have

$$f_{ij}^{(n)}(t) \geq 0 \quad (i, j = 0, 1, \ldots, n;\ t \geq 0), \tag{1·14}$$

$$f_{ij}^{(n)}(0) = \delta_{ij} \quad (i, j = 0, 1, \ldots, n). \tag{1·15}$$

Post-multiplying the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} F^{(n)}(t) = F^{(n)}(t) A^{(n)}$$

by the column vector

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

we find that
$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=0}^{n} f_{ij}^{(n)}(t) = -\lambda_n f_{in}^{(n)}(t) \leqslant 0$$
by (1·14). Since
$$\sum_{j=0}^{n} f_{ij}^{(n)}(0) = 1,$$
it follows that
$$\sum_{j=0}^{n} f_{ij}^{(n)}(t) \leqslant 1 \quad (i = 0, 1, \ldots, n;\ t \geqslant 0), \tag{1·16}$$
and in particular
$$0 \leqslant f_{ij}^{(n)}(t) \leqslant 1. \tag{1·16}'$$

Our aim is to obtain a spectral resolution of $F^{(n)}(t)$. But before we can apply (1·10) we have to show that the eigenvalues of $A^{(n)}$ are distinct. This leads to a discussion of the sequence of polynomials

$$\phi_n(\xi) = \begin{vmatrix} \xi+\lambda_0 & -\lambda_0 & & & & \\ -\mu_1 & \xi+\lambda_1+\mu_1 & -\lambda_1 & & & \\ & -\mu_2 & \xi+\lambda_2+\mu_2 & -\lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -\mu_{n-1} & \xi+\lambda_{n-1}+\mu_{n-1} & -\lambda_{n-1} \\ & & & & -\mu_n & \xi+\lambda_n+\mu_n \end{vmatrix} \tag{1·17}$$

($n = 1, 2, \ldots$). These polynomials play a fundamental part in the theory, and their properties have to be studied in some detail.

On expanding the determinant (1·17) with respect to the last row and column we obtain the recurrence relation
$$\phi_n(\xi) - (\xi+\lambda_n+\mu_n)\,\phi_{n-1}(\xi) + \lambda_{n-1}\mu_n\phi_{n-2}(\xi) = 0, \tag{1·18}$$
which is valid for $n \geqslant 1$ if we define
$$\phi_{-1}(\xi) = 1, \quad \phi_0(\xi) = \xi+\lambda_0. \tag{1·19}$$
We note that
$$\phi_1(\xi) = \xi^2 + (\lambda_0+\lambda_1+\mu_1)\,\xi + \lambda_0\lambda_1.$$
The roots of $\phi_n(\xi) = 0$ will be denoted by
$$\alpha_0^{(n)}, \alpha_1^{(n)}, \ldots, \alpha_n^{(n)},$$
so that
$$\phi_n(\xi) = \prod_{r=0}^{n} (\xi - \alpha_r^{(n)}).$$

The existence of the recurrence relation (1·18) enables us to establish a number of facts about the $\alpha_r^{(n)}$ which will be needed later. Our argument is based on the following:

Lemma 1. *Let $P_n(\xi)$ ($n = 1, 2, 3, \ldots$) be a sequence of polynomials defined by*
$$\left.\begin{aligned} &P_0(\xi) = 1, \quad P_1(\xi) = \xi + b_1, \\ &P_n(\xi) - (\xi + a_n)\,P_{n-1}(\xi) + b_n P_{n-2}(\xi) = 0 \quad (n = 2, 3, \ldots), \end{aligned}\right\} \tag{1·20}$$
where the a_n are real and the b_n are positive numbers such that
$$P_n(0) > 0. \tag{1·21}$$
Then the roots of $P_n(\xi) = 0$ ($n \geqslant 1$) are distinct and negative and are separated by the roots of
$$P_{n-1}(\xi) = 0.$$

Thus we may put $$P_n(\xi) = \prod_{r=1}^{n} (\xi - \gamma_r^{(n)}),$$

$$\left.\begin{aligned} &\text{where} && 0 > \gamma_1^{(n)} > \gamma_2^{(n)} > \ldots > \gamma_n^{(n)} \\ &\text{and} && 0 > \gamma_1^{(n)} > \gamma_1^{(n-1)} > \gamma_2^{(n)} > \gamma_2^{(n-1)} > \ldots > \gamma_{n-1}^{(n)} > \gamma_{n-1}^{(n-1)} > \gamma_n^{(n)}. \end{aligned}\right\} \tag{1·22}$$

Proof. We observe that (1·20) implies that $P_n(\xi)$ is a polynomial of degree n whose leading coefficient is unity. Since $b_1 > 0$, the only root of $P_1(\xi) = 0$ is negative. Using (1·21) and (1·20) for $n = 2$ we find that

$$P_2(0) > 0, \quad P_2(-b_1) = -b_2 < 0, \quad P_2(-\infty) > 0,$$

(where by the last statement we mean that $P_2(\xi)$ is positive for sufficiently great negative values of ξ). Thus each of the intervals

$$(-\infty, -b_1), \quad (-b_1, 0)$$

contains a root of $P_2(\xi) = 0$; and, since $-b_1 = \gamma_1^{(1)}$, we have

$$0 > \gamma_1^{(2)} > \gamma_1^{(1)} > \gamma_2^{(2)}.$$

Thus we have established a basis for induction.

Assuming now that $n \geqslant 3$ and that

$$0 > \gamma_1^{(n-1)} > \gamma_1^{(n-2)} > \ldots > \gamma_{n-2}^{(n-1)} > \gamma_{n-2}^{(n-2)} > \gamma_{n-1}^{(n-1)},$$

we deduce that $$\operatorname{sgn} P_{n-2}(\gamma_r^{(n-1)}) = (-1)^{r-1} \quad (r = 1, 2, \ldots, n-1). \tag{1·23}$$

On putting $\xi = \gamma_r^{(n-1)}$ in (1·20) we obtain that

$$P_n(\gamma_r^{(n-1)}) = -b_n P_{n-2}(\gamma_r^{(n-1)}),$$

whence $$\operatorname{sgn} P_n(\gamma_r^{(n-1)}) = (-1)^r \quad (r = 1, 2, \ldots, n-1).$$

Since $\operatorname{sgn} P_n(-\infty) = (-1)^n$, it follows that each of the intervals

$$(-\infty, \gamma_{n-1}^{(n-1)}),\ (\gamma_{n-1}^{(n-1)}, \gamma_{n-2}^{(n-1)}), \ldots,\ (\gamma_1^{(n-1)}, 0)$$

contains a root of $P_n(\xi) = 0$. This completes the proof of the lemma.

In applying the lemma to the polynomials $\phi_n(\xi)$ we have to distinguish two cases:

(i) $\lambda_0 > 0$. Put $$P_n(\xi) = \phi_{n-1}(\xi) \quad (n = 0, 1, 2, \ldots).$$

The recurrence relation (1·18), with $n+1$ instead of n, is of the type required in the lemma, and it remains to show that $\phi_n(0) > 0$. In fact,

$$\phi_n(0) = \lambda_0 \lambda_1 \ldots \lambda_n \quad (n = 0, 1, 2, \ldots). \tag{1·24}$$

For this is certainly true when n is equal to 1 or 2, and on putting $\xi = 0$ in (1·18) we get

$$\phi_n(0) = (\lambda_n + \mu_n)(\lambda_0 \lambda_1 \ldots \lambda_{n-1}) - \lambda_{n-1}\mu_n(\lambda_0\lambda_1 \ldots \lambda_{n-2}) = \lambda_0 \lambda_1 \ldots \lambda_n.$$

Hence by the lemma the roots of $\phi_n(\xi) = 0$ are distinct and negative and are separated by the roots of $\phi_{n-1}(\xi) = 0$.

(ii) $\lambda_0 = 0$. Equation (1·24) now shows that $\phi_n(0) = 0$ $(n = 0, 1, \ldots)$, so that we may put

$$\phi_n(\xi) = \xi \bar{\phi}_n(\xi). \tag{1·25}$$

The polynomials $\overline{\phi}_n(\xi)$ satisfy the same recurrence relations as the $\phi_n(\xi)$, viz.

$$\overline{\phi}_n(\xi)-(\xi+\lambda_n+\mu_n)\,\overline{\phi}_{n-1}(\xi)+\lambda_{n-1}\mu_n\overline{\phi}_{n-2}(\xi)=0. \tag{1·26}$$

In order to be able to apply lemma 1 to the polynomials we have to show that

$$\overline{\phi}_n(0)>0 \quad (n=0,1,2,\ldots).$$

By (1·26),

$$\overline{\phi}_n(0)-(\lambda_n+\mu_n)\,\overline{\phi}_{n-1}(0)+\lambda_{n-1}\mu_n\overline{\phi}_{n-2}(0)=0.$$

Put

$$z_0=1, \quad z_n=(\lambda_1\lambda_2\ldots\lambda_n)^{-1}\overline{\phi}_n(0). \tag{1·27}$$

The quantities z_n satisfy the recurrence relation

$$\lambda_n z_n-(\lambda_n+\mu_n)\,z_{n-1}+\lambda_{n-1}\mu_n z_{n-2}=0 \quad (n\geqslant 2).$$

This equation may be written

$$\lambda_n\Delta z_n-\mu_n\Delta z_{n-1}=0, \tag{1·28}$$

where $\Delta z_n=z_n-z_{n-1}$. On solving (1·28) recursively and noting that

$$\Delta z_1=z_1-z_0=\lambda_1^{-1}(\lambda_1+\mu_1)-1=\mu_1/\lambda_1,$$

we find that

$$\Delta z_n=\frac{\mu_1\mu_2\ldots\mu_n}{\lambda_1\lambda_2\ldots\lambda_n}.$$

As expressions of this kind will occur rather frequently in the sequel, it is convenient to use the following abbreviations:

$$\left.\begin{aligned} l_{-1}=\lambda_0, \quad l_0=1, \quad l_\nu&=(\lambda_1\lambda_2\ldots\lambda_\nu)^{-1} \quad (\nu\geqslant 1),\\ m_0=1, \quad m_\nu&=(\mu_1\mu_2\ldots\mu_\nu)^{-1} \quad (\nu\geqslant 1),\end{aligned}\right\} \tag{1·29}$$

and

$$w_0=1, \quad w_n=\frac{\lambda_1\lambda_2\ldots\lambda_n}{\mu_1\mu_2\ldots\mu_n}=\frac{m_n}{l_n} \quad (n\geqslant 1). \tag{1·30}$$

We can then write

$$z_n=\sum_{\nu=0}^{n} w_\nu^{-1},$$

and hence

$$\overline{\phi}_n(0)=l_n^{-1}\sum_{\nu=0}^{n} w_\nu^{-1}, \tag{1·31}$$

which makes it obvious that $\overline{\phi}_n(0)>0$.

Theorem 1. *The eigenvalues of $A^{(n)}$ are distinct and negative or zero, say*

$$0\geqslant\alpha_0^{(n)}>\alpha_1^{(n)}>\ldots>\alpha_n^{(n)}, \tag{1·32}$$

where

$$\alpha_0^{(n)}=0, \tag{1·33}$$

if and only if $\lambda_0=0$.

The eigenvalues of $A^{(n-1)}$ separate those of $A^{(n)}$, so that

$$0\geqslant\alpha_0^{(n)}\geqslant\alpha_0^{(n-1)}>\alpha_1^{(n)}>\alpha_1^{(n-1)}>\ldots>\alpha_{n-1}^{(n)}>\alpha_{n-1}^{(n-1)}>\alpha_n^{(n)}. \tag{1·34}$$

We are now justified in applying to $A^{(n)}$ the spectral resolution outlined in §1. For this purpose we have to find non-zero vectors $x_r^{(n)}$ and $y_r^{(n)}$ $(r=0,1,\ldots,n)$ satisfying

$$x_r^{(n)}(\alpha_r^{(n)}I^{(n)}-A^{(n)})=0, \tag{1·35}$$

$$(\alpha_r^{(n)}I^{(n)}-A^{(n)})\,y_r^{(n)\prime}=0, \tag{1·36}$$

respectively. The vectors need not be normalized so that we can take $x_r^{(n)}$ ($y_r^{(n)\prime}$) to be proportional to any non-zero row (column) of

$$\operatorname{adj}(\alpha_r^{(n)} I^{(n)} - A^{(n)}). \tag{1·37}$$

If $\lambda_0 > 0$, it is seen, after some elementary calculations, that the last row and column of (1·37) are non-zero. Their elements can be expressed in terms of the polynomials

$$\phi_\nu(\xi) \quad (\nu = 0, 1, 2, \ldots).$$

In fact, with a suitable choice of a factor we find that

$$x_r^{(n)} = (1, m_1\phi_0(\alpha_r^{(n)}), m_2\phi_1(\alpha_r^{(n)}), \ldots, m_n\phi_{n-1}(\alpha_r^{(n)})), \tag{1·38}$$

$$y_r^{(n)\prime} = (\lambda_0, l_0\phi_0(\alpha_r^{(n)}), l_1\phi_1(\alpha_r^{(n)}), \ldots, l_{n-1}\phi_{n-1}(\alpha_r^{(n)})) \tag{1·39}$$

$(r = 0, 1, \ldots, n)$.

If $\lambda_0 = 0$, these formulae can still be used except when $r = 0$, in which case $y_0^{(n)}$ would reduce to the zero vector. Since we now have $\alpha_0^{(n)} = 0$, the corresponding eigenvector, say $\bar{y}_0^{(n)\prime}$, satisfies the equation

$$A\bar{y}_0^{(n)\prime} = 0.$$

A non-zero solution is furnished by the first column of (1·37).

Let

$$\bar{y}_0^{(n)\prime} = (\zeta_0^{(n)}, \zeta_1^{(n)}, \ldots, \zeta_n^{(n)}). \tag{1·40}$$

It can be verified that we may put

$$\zeta_0^{(n)} = 1, \quad \zeta_r^{(n)} = 1 - \left(\sum_{\nu=0}^{r-1} w_\nu^{-1}\right)\left(\sum_{\nu=0}^{n} w_\nu^{-1}\right)^{-1} \tag{1·41}$$

$(r = 1, 2, \ldots, n)$.

Defining

$$\theta_r^{(n)} = x_r^{(n)} y_r^{(n)\prime} \quad (r = 0, 1, \ldots, n) \tag{1·42}$$

as in (1·4), we obtain

$$\left.\begin{aligned} &\theta_0^{(n)} = 1 \quad \text{if} \quad \lambda_0 = 0, \\ \text{and} \quad &\theta_r^{(n)} = \sum_{\nu=0}^{n} l_{\nu-1} m_\nu \{\phi_{\nu-1}(\alpha_r^{(n)})\}^2 \end{aligned}\right\} \tag{1·43}$$

in all other cases, i.e. for $0 \leqslant r \leqslant n$ when $\lambda_0 > 0$ and for $1 \leqslant r \leqslant n$ when $\lambda_0 = 0$.

The idempotents belonging to $A^{(n)}$ will be denoted by

$$D_r^{(n)} = y_r^{(n)\prime} x_r^{(n)} / \theta_r^{(n)} = (d_r^{(n)}(i, j)), \tag{1·44}$$

where we describe the position of a matrix element not by suffixes but by two integer variables i, j $(0 \leqslant i, j \leqslant n)$. For later reference it is necessary to write down explicit expressions for the matrices (1·44). Using (1·39), (1·40) and (1·41) we find that

$$d_0^{(n)}(i, j) = \zeta_i^{(n)} \delta_{0j} \quad \text{if} \quad \lambda_0 = 0, \tag{1·45}$$

and

$$d_r^{(n)}(i, j) = l_{i-1} m_j \phi_{i-1}(\alpha_r^{(n)}) \phi_{j-1}(\alpha_r^{(n)}) / \theta_r^{(n)} \tag{1·46}$$

in all other cases.

Finally, we arrive at the spectral resolution of the matrix $F^{(n)}(t)$, introduced in (1·13), namely,

$$F^{(n)}(t) = \sum_{r=0}^{n} \exp(t\alpha_r^{(n)}) D_r^{(n)}, \tag{1·47}$$

i.e.

$$f_{ij}^{(n)}(t) = \sum_{r=0}^{n} \exp(t\alpha_r^{(n)}) d_r^{(n)}(i, j) \tag{1·48}$$

$(i, j = 0, 1, \ldots, n)$.

3. *The modified sections*

As we have already indicated in the Introduction, a solution of the problem can be constructed by letting $n \to \infty$ in (1·48). This solution, obtained from the sequence of complete sections, plays a fundamental part in the theory of more general discrete Markov processes, as we have shown elsewhere (D). For certain types of simple birth-and-death processes it is possible to modify the standard procedure and to obtain a solution of a slightly different form. In many cases the two solutions can be proved to be identical, but there are circumstances (chapter III, §12) where they are distinct and where the problem has not a unique solution. This second solution is also obtained from a sequence of finite matrices, $G^{(n)}(t)$, which we are now going to define.

If, in (1·12), we change the last element from $-(\lambda_n+\mu_n)$ to $-\mu_n$, we obtain a matrix

$$B^{(n)} = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & & \\ & & \cdots & \cdots & \cdots & \\ & & & \mu_{n-1} & -(\lambda_{n-1}+\mu_{n-1}) & \lambda_{n-1} \\ & & & & \mu_n & -\mu_n \end{pmatrix} \tag{1·49}$$

which will be termed the nth *modified section*. As all row-sums of $B^{(n)}$ are zero, the matrix corresponds to a finite Markov process with $n+1$ states. The characteristic polynomial of $B^{(n)}$ will be denoted by $\psi_n(\xi)$ and the eigenvalues by

$$\beta_0^{(n)}, \beta_1^{(n)}, \ldots, \beta_n^{(n)}. \tag{1·50}$$

Thus
$$\psi_n(\xi) = |\xi I^{(n)} - B^{(n)}| = \prod_{r=0}^{n} (\xi - \beta_r^{(n)}) \tag{1·51}$$

$(n \geqslant 1)$. It is convenient to define

$$\psi_{-1}(\xi) = 0, \quad \psi_0(\xi) = \xi. \tag{1·52}$$

Since $B^{(n)}$ is a singular matrix, one of its eigenvalues is zero, and we shall put

$$\beta_0^{(n)} = 0 \quad (n \geqslant 0). \tag{1·53}$$

On expanding (1·17) with respect to the last row we get the identity

$$\phi_n(\xi) - \lambda_n \phi_{n-1}(\xi) = \psi_n(\xi) \tag{1·54}$$

$(n \geqslant 0)$. We shall now show that the polynomials $\psi_n(\xi)$ satisfy the recurrence relation

$$\psi_n(\xi) - (\xi + \lambda_{n-1} + \mu_n)\, \psi_{n-1}(\xi) + \lambda_{n-1}\mu_{n-1}\psi_{n-2}(\xi) = 0 \tag{1·55}$$

valid for $n \geqslant 1$, remembering that $\mu_0 = 0$. For (1·18) may be written (omitting the argument ξ)

$$(\phi_n - \lambda_n \phi_{n-1}) - \mu_n(\phi_{n-1} - \lambda_{n-1}\phi_{n-2}) = \xi\phi_{n-1}, \tag{1·56}$$

whence, by (1·54),
$$\psi_n - \mu_n \psi_{n-1} = \xi\phi_{n-1}. \tag{1·57}$$

In (1·54) replace n by $n-1$ and then multiply throughout by ξ, thus

$$\xi\phi_{n-1} - \xi\lambda_{n-1}\phi_{n-2} = \xi\psi_{n-1}.$$

On eliminating the ϕ's by means of (1·57) we obtain

$$(\psi_n-\mu_n\psi_{n-1})-\lambda_{n-1}(\psi_{n-1}-\mu_{n-1}\psi_{n-2})=\xi\psi_{n-1},$$

which is equivalent to (1·55).

In view of (1·53) we have, of course, $\psi_n(0)=0$ for all values of n, so that we may put

$$\psi_n(\xi)=\xi\bar{\psi}_n(\xi). \tag{1·58}$$

The polynomials $\bar{\psi}_n(\xi)$ satisfy the same recurrence relation as the $\psi_n(\xi)$, viz.

$$\bar{\psi}_n(\xi)-(\xi+\lambda_{n-1}+\mu_n)\,\bar{\psi}_{n-1}(\xi)+\lambda_{n-1}\mu_{n-1}\bar{\psi}_{n-2}(\xi)=0 \tag{1·59}$$

($n\geqslant 1$), and in accordance with (1·52) we have

$$\bar{\psi}_{-1}(\xi)=0,\quad \bar{\psi}_0(\xi)=1. \tag{1·60}$$

In order to be able to apply lemma 1 to the $\bar{\psi}_n(\xi)$ we have to show that

$$\psi_n'(0)=\bar{\psi}_n(0)>0\quad (n=0,1,2,\ldots). \tag{1·61}$$

On differentiating (1·57) and putting $\xi=0$ we find that

$$\psi_n'(0)=\mu_n\psi_{n-1}'(0)+\phi_{n-1}(0), \tag{1·62}$$

whence (1·61) follows by induction, because $\phi_{n-1}(0)>0$ by (1·24) and $\psi_1'(0)>0$.

Bearing in mind that

$$\bar{\psi}_n(\xi)=\prod_{r=1}^{n}(\xi-\beta_r^{(n)})\quad (n=1,2,\ldots), \tag{1·63}$$

we obtain the following facts about the eigenvalues of the modified sections:

THEOREM 2. *The eigenvalues of $B^{(n)}$ $(n=1,2,\ldots)$ are distinct; one of them is zero and the others negative, thus*

$$0=\beta_0^{(n)}>\beta_1^{(n)}>\ldots>\beta_n^{(n)}. \tag{1·64}$$

The eigenvalues of $B^{(n-1)}$ separate those of $B^{(n)}$, i.e.

$$0=\beta_0^{(n)}=\beta_0^{(n-1)}>\beta_1^{(n)}>\beta_1^{(n-1)}>\ldots>\beta_{n-1}^{(n)}>\beta_{n-1}^{(n-1)}>\beta_n^{(n)}. \tag{1·65}$$

The spectral theory of §1 may be applied to $B^{(n)}$. The eigenvectors $u_r^{(n)}$ and $v_r^{(n)}$ corresponding to $\beta_r^{(n)}$ may be taken to be any non-zero row or column of

$$\operatorname{adj}(\beta_r^{(n)}I^{(n)}-B^{(n)}) \tag{1·66}$$

respectively. Since $A^{(n)}$ and $B^{(n)}$ differ only in their last element, the cofactors of the last row and of the last column are common to

$$|\xi I^{(n)}-A^{(n)}|\quad\text{and}\quad|\xi I^{(n)}-B^{(n)}|. \tag{1·67}$$

Apart from $v_0^{(n)}$ when $\lambda_0=0$, the eigenvectors of $B^{(n)}$ can be obtained from (1·38) and (1·39) simply by replacing $\alpha_r^{(n)}$ by $\beta_r^{(n)}$. Thus

$$\left.\begin{aligned}u_r^{(n)}=(1,m_1\phi_0(\beta_r^{(n)}),m_2\phi_1(\beta_r^{(n)}),\ldots,m_n\phi_{n-1}(\beta_r^{(n)}))\\(0\leqslant r\leqslant n;\ \lambda_0\geqslant 0),\end{aligned}\right\} \tag{1·68}$$

$$\left.\begin{aligned}v_r^{(n)}=(\lambda_0,l_0\phi_0(\beta_r^{(n)}),l_1\phi_1(\beta_r^{(n)}),\ldots,l_{n-1}\phi_{n-1}(\beta_r^{(n)}))\\(0\leqslant r\leqslant n\text{ if }\lambda_0>0,\text{ and }1\leqslant r\leqslant n\text{ if }\lambda_0=0).\end{aligned}\right\} \tag{1·69}$$

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Hence
$$\left.\begin{array}{l}\eta_r^{(n)} = u_r^{(n)} v_r^{(n)\prime} = \lambda_0 + \sum_{\nu=1}^{n} l_{\nu-1} m_\nu \{\phi_{\nu-1}(\beta_r^{(n)})\}^2 \\ (0 \leqslant r \leqslant n \text{ if } \lambda_0 > 0, \text{ and } 1 \leqslant r \leqslant n \text{ if } \lambda_0 = 0).\end{array}\right\} \tag{1·70}$$

We note in particular that
$$\eta_0^{(n)} = \lambda_0 + \lambda_0^2 \sum_{\nu=1}^{n} w_{\nu-1}/\mu_\nu \quad (\lambda_0 > 0), \tag{1·71}$$
because
$$\phi_{\nu-1}(\beta_0^{(n)}) = \phi_{\nu-1}(0) = \lambda_0/l_{\nu-1}.$$
When $\lambda_0 = 0$ we cannot use (1·69) for $r = 0$ as this would yield a zero vector. Instead we take
$$v_0^{(n)} = (1, 1, \ldots, 1) \quad (\lambda_0 = 0),$$
which obviously satisfies $B^{(n)} v_0^{(n)\prime} = 0$, and we now get
$$\eta_0^{(n)} = u_0^{(n)} v_0^{(n)\prime} = 1 \quad (\lambda_0 = 0), \tag{1·72}$$
because
$$u_0^{(n)} = (1, 0, 0, \ldots, 0) \tag{1·73}$$
if $\lambda_0 = 0$.

The idempotents of $B^{(n)}$ will be denoted by
$$E_r^{(n)} = v_r^{(n)\prime} u_r^{(n)} / \eta_r^{(n)} = (e_r^{(n)}(i, j)), \tag{1·74}$$
where
$$\left.\begin{array}{l} e_r^{(n)}(i, j) = l_{i-1} m_j \phi_{i-1}(\beta_r^{(n)}) \, \phi_{j-1}(\beta_r^{(n)}) / \eta_r^{(n)} \\ (0 \leqslant r \leqslant n \text{ if } \lambda_0 > 0, \text{ and } 1 \leqslant r \leqslant n \text{ if } \lambda_0 = 0)\end{array}\right\} \tag{1·75}$$
and
$$e_0^{(n)}(i, j) = \delta_{0j} \quad (\lambda_0 = 0). \tag{1·76}$$

In order to construct a solution from the modified sections we define
$$G^{(n)}(t) = (g_{ij}^{(n)}(t))$$
as the solution of
$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = Y(t) \, B^{(n)} \tag{1·77}$$
satisfying the initial conditions $Y(0) = I^{(n)}$. Thus
$$G^{(n)}(t) = \exp(tB^{(n)}) = \sum_{r=0}^{n} \exp(t\beta_r^{(n)}) \, E_r^{(n)}, \tag{1·78}$$
i.e.
$$g_{ij}^{(n)}(t) = \sum_{r=0}^{n} \exp(t\beta_r^{(n)}) \, e_r^{(n)}(i, j). \tag{1·79}$$
In analogy with (1·14), (1·15) and (1·16) we now have
$$g_{ij}^{(n)}(t) \geqslant 0 \quad (i, j = 0, 1, \ldots, n; \; t \geqslant 0), \tag{1·80}$$
$$g_{ij}^{(n)}(0) = \delta_{ij} \quad (i, j = 0, 1, \ldots, n), \tag{1·81}$$
$$\sum_{j=0}^{n} g_{ij}^{(n)}(t) = 1 \quad (i = 0, 1, \ldots, n; \; t \geqslant 0), \tag{1·82}$$
no inequality sign occurring in (1·82) since all row-sums of $B^{(n)}$ are zero.

From the expressions as exponentials it follows at once that $F^{(n)}(t)$ and $G^{(n)}(t)$ satisfy the Chapman-Kolmogorov equation
$$F^{(n)}(t+\tau) = F^{(n)}(t) \, F^{(n)}(\tau), \tag{1·83}$$
$$G^{(n)}(t+\tau) = G^{(n)}(t) \, G^{(n)}(\tau), \tag{1·84}$$

i.e. explicitly
$$f_{ij}^{(n)}(t+\tau) = \sum_{k=0}^{n} f_{ik}^{(n)}(t)\, f_{kj}^{(n)}(\tau), \tag{1·83'}$$
$$(0 \leqslant i, j \leqslant n).$$
$$g_{ij}^{(n)}(t+\tau) = \sum_{k=0}^{n} g_{ik}^{(n)}(t)\, g_{kj}^{(n)}(\tau). \tag{1·84'}$$

Since a finite matrix Z commutes with $\exp Z$, it follows that $F^{(n)}(t)$ and $G^{(n)}(t)$ not only satisfy (1·11) and (1·78), but also
$$\frac{\mathrm{d}}{\mathrm{d}t} F^{(n)}(t) = A^{(n)} F^{(n)}(t) \tag{1·85}$$
and
$$\frac{\mathrm{d}}{\mathrm{d}t} G^{(n)}(t) = B^{(n)} G^{(n)}(t), \tag{1·86}$$
respectively. These equations, which play an important part in the general theory of Markov processes, are called the *backward equations*.

We conclude this section by proving some inequalities for the eigenvalues of $A^{(n)}$ and $B^{(n)}$, and for the matrices $F^{(n)}(t)$ and $G^{(n)}(t)$.

THEOREM 3. *In the notation of theorems* 1 *and* 2,
$$0 \geqslant \beta_r^{(n)} \geqslant \alpha_r^{(n)} \quad (r = 0, 1, \ldots, n). \tag{1·87}$$

Proof. First, let $\lambda_0 > 0$. Putting $\xi = \alpha_r^{(n)}$ in (1·54) we have
$$\psi_n(\alpha_r^{(n)}) = -\lambda_n \phi_{n-1}(\alpha_r^{(n)}),$$
whence by (1·34)
$$\operatorname{sgn} \psi_n(\alpha_r^{(n)}) = -\operatorname{sgn} \phi_{n-1}(\alpha_r^{(n)}) = (-1)^{r+1}$$
$(r = 0, 1, \ldots, n)$. It follows that each of the intervals
$$(\alpha_1^{(n)}, \alpha_0^{(n)}),\ (\alpha_2^{(n)}, \alpha_1^{(n)}),\ \ldots,\ (\alpha_n^{(n)}, \alpha_{n-1}^{(n)})$$
contains a (non-zero) root of $\psi_n(\xi) = 0$ so that
$$\alpha_0^{(n)} > \beta_1^{(n)} > \alpha_1^{(n)} > \beta_2^{(n)} > \alpha_2^{(n)} > \ldots > \beta_n^{(n)} > \alpha_n^{(n)}.$$
This, together with
$$0 = \beta_0^{(n)} > \alpha_0^{(n)},$$
proves (1·87).

Next, if $\lambda_0 = 0$, we divide (1·54) throughout by ξ and, in the notation of (1·25) and (1·58), we get
$$\bar{\phi}_n(\xi) - \lambda_n \phi_{n-1}(\xi) = \bar{\psi}_n(\xi).$$
The above argument may be applied to the functions $\bar{\phi}_{n-1}(\xi)$ and $\bar{\psi}_n(\xi)$ yielding the result that
$$0 > \beta_1^{(n)} > \alpha_1^{(n)} > \ldots > \beta_n^{(n)} > \alpha_n^{(n)}.$$
Since we now have $0 = \beta_0^{(n)} = \alpha_0^{(n)}$, the theorem is proved also in this case.

THEOREM 4. *For fixed* t $(\geqslant 0)$,
$$g_{ij}^{(n)}(t) \geqslant f_{ij}^{(n)}(t) \quad (0 \leqslant i, j \leqslant n) \tag{1·88}$$
and
$$f_{ij}^{(n)}(t) \geqslant f_{ij}^{(n-1)}(t) \quad (0 \leqslant i, j \leqslant n-1). \tag{1·89}$$

Proof (see D, lemma 4). If $\quad U = (u_{ij}), \quad V = (v_{ij})$

are matrices of the same order we use the notation
$$U \geqslant V$$

to express that
$$u_{ij} \geqslant v_{ij}$$
for all i and j. Clearly,
$$U \geqslant V \geqslant 0$$
implies that
$$\exp U \geqslant \exp V.$$
As on p. 327, let
$$q = \max(\lambda_0, \lambda_1+\mu_1, \ldots, \lambda_n+\mu_n)$$
and put
$$U = A^{(n)}+qI^{(n)}, \quad V = B^{(n)}+qI^{(n)}.$$
Evidently, by (1·12) and (1·49),
$$V \geqslant U \geqslant 0.$$
Hence, if $t \geqslant 0$,
$$\exp\{t(B^{(n)}+qI^{(n)})\} \geqslant \exp\{t(A^{(n)}+qI^{(n)})\},$$
$$e^{tq}\exp(tB^{(n)}) \geqslant e^{tq}\exp(tA^{(n)}),$$
and therefore
$$G^{(n)}(t) \geqslant F^{(n)}(t).$$
In order to prove (1·89) we introduce the matrices
$$A_0^{(n)} = \begin{pmatrix} A^{(n-1)} & 0 \\ 0 & -(\lambda_n+\mu_n) \end{pmatrix}$$
and
$$F_0^{(n)}(t) = \exp(tA_0^{(n)}).$$
Since
$$A^{(n)}+qI^{(n)} \geqslant A_0^{(n)}+qI^{(n)} \geqslant 0,$$
we deduce that, for $t \geqslant 0$,
$$e^{tq}\exp(tA^{(n)}) \geqslant e^{tq}\exp(tA_0^{(n)}),$$
and hence
$$F^{(n)}(t) \geqslant F_0^{(n)}(t).$$
On restricting this result to the first n rows and columns we obtain (1·89).

Chapter II. Spectral theory for birth and death processes

4. *The spectral resolution* ($\lambda_0 > 0$)

The formulae of the preceding chapter, relating to the spectral resolutions for the finite sections $A^{(n)}$, $B^{(n)}$ and for the corresponding functions $f_{ij}^{(n)}(t)$, $g_{ij}^{(n)}(t)$, will now be used to construct solutions $f_{ij}(t)$, $g_{ij}(t)$ of the infinite system of equations (0·14). This will be done by means of a limiting process, whose discussion is facilitated by writing the spectral resolution in Stieltjes integral form. We shall first treat the case $\lambda_0 > 0$; the modifications needed when $\lambda_0 = 0$ will be dealt with in § 5.

For the section $A^{(n)}$, we define a 'spectral function' $\rho^{(n)}(x)$ as follows: $\rho^{(n)}(x)$ is to be a non-decreasing step-function, defined for $-\infty < x < \infty$, with discontinuities $1/\theta_r^{(n)}$ at
$$x = \alpha_r^{(n)} \quad (r = 0, 1, \ldots, n) \quad \text{and} \quad \rho^{(n)}(0) = 0.$$
Explicitly, $\rho^{(n)}(x)$ is given by
$$\rho^{(n)}(x) = \begin{cases} 0, & x \geqslant \alpha_0^{(n)}; \\ -\left(\dfrac{1}{\theta_0^{(n)}}+\dfrac{1}{\theta_1^{(n)}}+\ldots+\dfrac{1}{\theta_r^{(n)}}\right), & \alpha_{r+1}^{(n)} \leqslant x < \alpha_r^{(n)} \quad (r = 0, \ldots, n-1); \\ -\left(\dfrac{1}{\theta_0^{(n)}}+\ldots+\dfrac{1}{\theta_n^{(n)}}\right), & x < \alpha_n^{(n)}. \end{cases} \tag{2·1}$$

Observe that $\rho^{(n)}(x) = 0$ for $x \geqslant 0$, because $0 > \alpha_0^{(n)}$. From (1·46) and (1·48) it follows that

$$f_{ij}^{(n)}(t) = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, e^{tx}\, d\rho^{(n)}(x) \quad (0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n). \tag{2·2}$$

The integral (like all integrals with unspecified limits in this chapter) is taken from $-\infty$ to $+\infty$, i.e. effectively from $-\infty$ to 0, $\rho^{(n)}(x)$ being constant for $x \geqslant 0$. In particular, since $f_{ij}^{(n)}(0) = \delta_{ij}$, it follows by putting $t = 0$ in (2·2) that

$$l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, d\rho^{(n)}(x) = \delta_{ij} \quad (0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n). \tag{2·3}$$

Similar formulae can be developed for the sections $B^{(n)}$, the spectral function $\sigma^{(n)}(x)$ having discontinuities $1/\eta_r^{(n)}$ at $x = \beta_r^{(n)}$. Explicitly,

$$\sigma^{(n)}(x) = \left\{ \begin{array}{lll} 0 & x \geqslant \beta_0^{(n)}; & \\ -\left(\dfrac{1}{\eta_0^{(n)}} + \dfrac{1}{\eta_1^{(n)}} + \ldots + \dfrac{1}{\eta_r^{(n)}}\right), & \beta_{r+1}^{(n)} \leqslant x < \beta_r^{(n)} & (r = 0, \ldots, n-1); \\ -\left(\dfrac{1}{\eta_0^{(n)}} + \ldots + \dfrac{1}{\eta_n^{(n)}}\right), & x < \beta_n^{(n)}. & \end{array} \right\} \tag{2·4}$$

Then, by (1·75) and (1·79),

$$g_{ij}^{(n)}(t) = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, e^{tx}\, d\sigma^{(n)}(x) \quad (0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n), \tag{2·5}$$

and since $g_{ij}^{(n)}(0) = \delta_{ij}$,

$$l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, d\sigma^{(n)}(x) = \delta_{ij} \quad (0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n). \tag{2·6}$$

We may also observe from (1·6) that the sections $A^{(n)}$, $B^{(n)}$ themselves have spectral resolutions given by

$$A^{(n)} = \sum_{r=0}^{n} \alpha_r^{(n)} D_r^{(n)}, \quad B^{(n)} = \sum_{r=0}^{n} \beta_r^{(n)} E_r^{(n)}.$$

These two results, when expressed in Stieltjes integral form, read

$$\left. \begin{array}{l} a_{ij}^{(n)} = l_{i-1} m_j \displaystyle\int x \phi_{i-1}(x)\, \phi_{j-1}(x)\, d\rho^{(n)}(x) \\ b_{ij}^{(n)} = l_{i-1} m_j \displaystyle\int x \phi_{i-1}(x)\, \phi_{j-1}(x)\, d\sigma^{(n)}(x) \end{array} \right\} \quad (0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n); \tag{2·7}$$

they will not be used in this paper but are stated here for the sake of completeness.

The limiting process to be applied to $f_{ij}^{(n)}(t)$ and $g_{ij}^{(n)}(t)$ is based on some well-known results of Helly (1921), which are collected together in

LEMMA H. *Let a sequence of non-decreasing functions* $\rho^{(n)}(x)$ *be defined for* $-\infty < x < \infty$ *and uniformly bounded, i.e.*

$$|\rho^{(n)}(x)| < K, \tag{2·8}$$

K being independent of x, n. *Then there exist a non-decreasing function* $\rho(x)$, *and a subsequence* $\{\rho^{(n_k)}(x)\}$ *of* $\{\rho^{(n)}(x)\}$, *such that*

$$\rho^{(n_k)}(x) \to \rho(x) \quad \text{as} \quad n_k \to \infty \tag{2·9}$$

at all points of continuity of $\rho(x)$. *Moreover, if* $f(x)$ *is continuous for* $-\infty < x \leqslant 0$, *then*

$$\int_{-\infty}^{0} f(x)\, \mathrm{d}\rho^{(n_k)}(x) \to \int_{-\infty}^{0} f(x)\, \mathrm{d}\rho(x), \tag{2·10}$$

provided that the infinite integral on the left in (2·10) *converges, uniformly with respect to* n_k; *that is to say, if, given* $\epsilon > 0$, *there exist* $X(\epsilon)$ *and* $N(\epsilon)$ *such that*

$$\left| \int_{-\infty}^{-X(\epsilon)} f(x)\, \mathrm{d}\rho^{(n_k)}(x) \right| < \epsilon \quad \text{for} \quad n_k > N(\epsilon). \tag{2·11}$$

In order to apply this lemma, we must check that $\rho^{(n)}(x)$ and $\sigma^{(n)}(x)$, given by (2·1) and (2·4), satisfy (2·8); since $\rho^{(n)}(0) = \sigma^{(n)}(0) = 0$, we have only to show that

$$\int \mathrm{d}\rho^{(n)}(x), \quad \int \mathrm{d}\sigma^{(n)}(x)$$

are uniformly bounded. However, in order to verify the condition (2·11) we shall need the more general fact that

$$\int |x|^m \mathrm{d}\rho^{(n)}(x), \quad \int |x|^m \mathrm{d}\sigma^{(n)}(x) \quad (m = 0, 1, 2, \ldots)$$

are uniformly bounded (for each fixed m). This we prove by observing that $\phi_{s-1}(x)$ has degree s and leading term x^s, and hence that any power x^m can be expressed uniquely as a linear combination

$$x^m = \sum_{s=0}^{m} c_{ms} \phi_{s-1}(x).$$

Hence, for $n \geqslant m$,

$$\int x^m \mathrm{d}\rho^{(n)}(x) = \sum_{s=0}^{m} c_{ms} \int \phi_{s-1}(x)\, \mathrm{d}\rho^{(n)}(x)$$

$$= \sum_{s=0}^{m} c_{ms} \int \phi_{s-1}(x)\, \phi_{-1}(x)\, \mathrm{d}\rho^{(n)}(x) = \sum_{s=0}^{m} c_{ms} \frac{\delta_{s0}}{l_{s-1} m_0} = \frac{c_{m0}}{\lambda_0},$$

on using (2·3) with $i = s, j = 0$. Thus

$$\int_{-\infty}^{0} |x|^m \mathrm{d}\rho^{(n)}(x) = K_m = \frac{|c_{m0}|}{\lambda_0} \quad (n \geqslant m), \tag{2·12}$$

and by the same argument

$$\int_{-\infty}^{0} |x|^m \mathrm{d}\sigma^{(n)}(x) = K_m \quad (n \geqslant m). \tag{2·13}$$

We can now, using (2·12) with $m = 0$, select a subsequence $\rho^{(n_k)}(x)$ of $\rho^{(n)}(x)$, with limiting function $\rho(x)$ as in lemma H. We assert that, for any polynomial $P(x)$,

$$\int_{-\infty}^{0} P(x)\, \mathrm{e}^{tx}\, \mathrm{d}\rho^{(n_k)}(x) \to \int_{-\infty}^{0} P(x)\, \mathrm{e}^{tx}\, \mathrm{d}\rho(x) \quad \text{as} \quad n_k \to \infty \quad (t \geqslant 0).$$

It suffices to prove this when $P(x) = x^m$, and by Lemma H we need only verify condition (2·11) (with $f(x) = x^m \mathrm{e}^{tx}, t \geqslant 0$). Now

$$I(X) = \left| \int_{-\infty}^{-X} x^m \mathrm{e}^{tx}\, \mathrm{d}\rho^{(n_k)}(x) \right| \leqslant \int_{-\infty}^{-X} |x|^m \mathrm{d}\rho^{(n_k)}(x)$$

$$\leqslant \frac{1}{X} \int_{-\infty}^{-X} |x|^{m+1} \mathrm{d}\rho^{(n_k)}(x) \leqslant \frac{1}{X} \int_{-\infty}^{0} |x|^{m+1} \mathrm{d}\rho^{(n_k)}(x) = \frac{K_{m+1}}{X},$$

for $n_k \geqslant m+1$ (using (2·12)). Hence $I(X) < \epsilon$ for $X = X(\epsilon) > \frac{\epsilon}{K_{m+1}}$ and $n_k > N(\epsilon) = m$, and the assertion follows from lemma H. It now follows from (2·2) that $f_{ij}^{(n_k)}(t)$ tends, as $n_k \to \infty$, to a limit $f_{ij}(t)$ given by

$$f_{ij}(t) = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{e}^{tx}\, \mathrm{d}\rho(x), \tag{2·14}$$

and also that

$$f_{ij}(0) = \delta_{ij} = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{d}\rho(x). \tag{2·15}$$

Moreover, (2·14) may be differentiated under the integral sign (the differentiated integral being uniformly convergent for $t \geqslant 0$), and hence $f'_{ij}(t)$ exists and is given by

$$f'_{ij}(t) = l_{i-1} m_j \int x\phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{e}^{tx}\, \mathrm{d}\rho(x).$$

But $f_{ij}^{(n_k)\prime}(t)$ is given by a similar expression with $\rho^{(n_k)}(x)$ instead of $\rho(x)$, and it follows that

$$f'_{ij}(t) = \lim_{n_k \to \infty} f_{ij}^{(n_k)\prime}(t).$$

From the equation

$$F^{(n_k)\prime}(t) = F^{(n_k)}(t)\, A^{(n_k)},$$

retaining only the components $f_{ij}^{(n_k)}(t)$ with $0 \leqslant i < n_k$, $0 \leqslant j < n_k$, and from the form (1·12) of $A^{(n_k)}$, we see that

$$f_{ij}^{(n_k)\prime}(t) = \lambda_{j-1} f_{i,j-1}^{(n_k)}(t) - (\lambda_j + \mu_j) f_{ij}^{(n_k)}(t) + \mu_{j+1} f_{i,j+1}^{(n_k)}(t) \quad (0 \leqslant i < n_k, 0 \leqslant j < n_k)$$

(with the convention (0·3) that $\lambda_{-1} = \mu_0 = 0$). If we now let $n_k \to \infty$, it follows that

$$f'_{ij}(t) = \lambda_{j-1} f_{i,j-1}(t) - (\lambda_j + \mu_j) f_{ij}(t) + \mu_{j+1} f_{i,j+1}(t), \tag{2·16}$$

so that $p_{ij} = f_{ij}(t)$ is a solution of the infinite system (0·14). It is also clear (from (1·14) and (1·15)) that $f_{ij}(t) \geqslant 0$ for $t \geqslant 0$, and $f_{ij}(0) = \delta_{ij}$. Finally, from (1·16) we can easily deduce that

$$\sum_{j=0}^{\infty} f_{ij}(t) \leqslant 1. \tag{2·17}$$

For, taking a fixed J, and $n_k \geqslant J$, we have

$$\sum_{j=0}^{J} f_{ij}^{(n_k)}(t) \leqslant \sum_{j=0}^{n_k} f_{ij}^{(n_k)}(t) \leqslant 1.$$

Let $n_k \to \infty$, then we obtain

$$\sum_{j=0}^{J} f_{ij}(t) \leqslant 1,$$

and (2·17) follows on letting $J \to \infty$.

The whole argument can of course be repeated for $g_{ij}^{(n)}(t)$, choosing a convergent subsequence $\sigma^{(n_k)}(x)$ of $\sigma^{(n)}(x)$ with limit function $\sigma(x)$. In this way we find that $g_{ij}^{(n_k)}(t)$ tends to a limit $g_{ij}(t)$, and that

$$g_{ij}(t) = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{e}^{tx}\, \mathrm{d}\sigma(x), \tag{2·18}$$

$$g_{ij}(0) = \delta_{ij} = l_{i-1} m_j \int \phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{d}\sigma(x), \tag{2·19}$$

$$g'_{ij}(t) = \lambda_{j-1} g_{i,j-1}(t) - (\lambda_j + \mu_j)\, g_{ij}(t) + \mu_{j+1} g_{i,j+1}(t); \tag{2·20}$$

also $g_{ij}(t) \geqslant 0$ for $t \geqslant 0$, $g_{ij}(0) = \delta_{ij}$, and

$$\sum_{j=0}^{\infty} g_{ij}(t) \leqslant 1. \tag{2·21}$$

We mention finally that from (2·7) we can obtain 'spectral resolutions' for the infinite matrix A in terms of the spectral functions $\rho(x)$ and $\sigma(x)$, viz.

$$a_{ij} = l_{i-1} m_j \int x\phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{d}\rho(x) = l_{i-1} m_j \int x\phi_{i-1}(x)\, \phi_{j-1}(x)\, \mathrm{d}\sigma(x). \tag{2·22}$$

The existence of such spectral resolutions could be deduced from the theory of infinite Jacobi matrices (Stone 1932). (The matrix A is not a Jacobi matrix as it stands, since it is not symmetric, but can be transformed into a Jacobi matrix by multiplying rows and columns by suitable factors.) Indeed, some of the formal machinery of chapter I, and the 'selection' arguments of the present chapter, are familiar tools in the theory of Jacobi matrices and of the moment problem (cf. Stone 1932).

Further information about the solutions $f_{ij}(t)$, $g_{ij}(t)$ can be deduced from theorem 4. First, since $f_{ij}^{(n)}(t) \leqslant g_{ij}^{(n)}(t)$, it follows that (cf. D, theorem 3)

$$f_{ij}(t) \leqslant g_{ij}(t) \quad (t \geqslant 0). \tag{2·23}$$

We are unable to give an example where strict inequality occurs in (2·23), but there is little doubt that such examples exist, since we shall be able to give examples of the corresponding phenomenon when $\lambda_0 = 0$. Secondly, since $f_{ij}^{(n)}(t) \geqslant f_{ij}^{(n-1)}(t)$, it follows that, as $n \to \infty$, the sequence $f_{ij}^{(n)}(t)$ converges as it stands to a limit $f_{ij}(t)$, and hence any subsequence $f_{ij}^{(n_k)}(t)$ converges to this same limit. It follows that the sequence $\rho^{(n)}(x)$ must also converge as it stands (no selection being required); for, if not, there would exist two subsequences with distinct limit functions, $\rho(x)$ and $\rho^*(x)$. But, from (2·14) with $i = j = 0$, we should then have

$$f_{00}(t) = \lambda_0 \int e^{tx}\, \mathrm{d}\rho(x) = \lambda_0 \int e^{tx}\, \mathrm{d}\rho^*(x),$$

and by the uniqueness theorem for Laplace-Stieltjes integrals (Widder 1941, p. 63) $\rho(x)$ and $\rho^*(x)$ must after all be identical. We cannot, in general, assert that the sequences $g_{ij}^{(n)}(t)$ and $\sigma^{(n)}(x)$ converge as they stand, though we shall show in chapter III that they do so if restrictions are imposed on the behaviour of the coefficients λ_j and μ_j (as $j \to \infty$). A further consequence of the monotone convergence of $f_{ij}^{(n)}(t)$ to $f_{ij}(t)$ is, by an easy deduction (D, theorem 1) from (1·83′), that $f_{ij}(t)$ satisfies the Chapman-Kolmogorov equation

$$f_{ij}(t+\tau) = \sum_{\nu=0}^{\infty} f_{i\nu}(t) f_{\nu j}(\tau). \tag{2·24}$$

We can prove that $g_{ij}(t)$ satisfies (2·24) only under further assumptions on λ_j and μ_j.

5. *The spectral resolution* ($\lambda_0 = 0$)

The analysis of §4 must be modified when $\lambda_0 = 0$. The difficulty is that, while formulae similar to (2·2) and (2·5) still hold (at any rate if $i \geqslant 1$ and $j \geqslant 1$), the functions $\rho^{(n)}(x)$ and $\sigma^{(n)}(x)$ are, in general, not uniformly bounded so that the limiting process based on lemma H can no longer be applied. The difficulty can be surmounted by modifying the definition of the spectral functions. When $\lambda_0 = 0$, the polynomials $\phi_{s-1}(x)$ $(s \geqslant 1)$ are of the form

$$\phi_{s-1}(x) = x\psi_{s-1}(x),$$

so that a formula analogous to (2·2) would involve (when $i \geqslant 1, j \geqslant 1$) an integral of the form

$$\int x^2 \phi_{i-1}(x)\, \bar{\phi}_{j-1}(x)\, e^{tx}\, d\rho^{(n)}(x).$$

We shall find that if we remove a factor x and replace the above integral by

$$\int x \bar{\phi}_{i-1}(x)\, \bar{\phi}_{j-1}(x)\, e^{tx}\, d\bar{\rho}^{(n)}(x),$$

the modified spectral function $\bar{\rho}^{(n)}(x)$ (and the analogous $\bar{\sigma}^{(n)}(x)$) does remain uniformly bounded. We shall also find that, as might be expected, modifications are needed when $i = 0$ or $j = 0$ because of the changes in the expressions for $d_0^{(n)}(i,j)$ and $e_0^{(n)}(i,j)$ when $\lambda_0 = 0$ (cf. (1·45) and (1·76)).

We start by rewriting the expression (1·46) for $d_r^{(n)}(i,j)$ (valid for $1 \leqslant r \leqslant n$ when $\lambda_0 = 0$) as follows:

(i) If $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, then

$$d_r^{(n)}(i,j) = l_{i-1} m_j (\alpha_r^{(n)})^2\, \phi_{i-1}(\alpha_r^{(n)})\, \bar{\phi}_{j-1}(\alpha_r^{(n)}) / \theta_r^{(n)}.$$

Defining $\bar{\theta}_r^{(n)}$ by
$$\bar{\theta}_r^{(n)} = \theta_r^{(n)} / |\alpha_r^{(n)}| \quad (r = 1, \ldots, n), \tag{2·25}$$

we have
$$d_r^{(n)}(i,j) = -l_{i-1} m_j \alpha_r^{(n)} \phi_{i-1}(\alpha_r^{(n)})\, \bar{\phi}_{j-1}(\alpha_r^{(n)}) / \bar{\theta}_r^{(n)}. \tag{2·26}$$

(ii) If $i = 0$, then
$$d_r^{(n)}(i,j) = 0, \tag{2·27}$$

because $l_{i-1} = l_{-1} = \lambda_0 = 0$.

(iii) If $1 \leqslant i \leqslant n, j = 0$, then

$$\begin{aligned} d_r^{(n)}(i,j) &= l_{i-1} \alpha_r^{(n)} \bar{\phi}_{i-1}(\alpha_r^{(n)}) / \theta_r^{(n)} \\ &= -l_{i-1} \phi_{i-1}(\alpha_r^{(n)}) / \bar{\theta}_r^{(n)}. \end{aligned} \tag{2·28}$$

Now define $\bar{\rho}^{(n)}(x)$ to have discontinuities $1/\bar{\theta}_r^{(n)}$ at $x = \alpha_r^{(n)}$ for $r = 1, 2, \ldots, n$ (note that $r = 0$ is now excluded), and to have $\bar{\rho}^{(n)}(0) = 0$; explicitly

$$\bar{\rho}^{(n)}(x) = \begin{cases} 0, & x \geqslant \alpha_1^{(n)}; \\ -\left(\dfrac{1}{\bar{\theta}_1^{(n)}} + \ldots + \dfrac{1}{\bar{\theta}_r^{(n)}}\right), & \alpha_{r+1}^{(n)} \leqslant x < \alpha_r^{(n)} \quad (r = 1, \ldots, n-1); \\ -\left(\dfrac{1}{\bar{\theta}_1^{(n)}} + \ldots + \dfrac{1}{\bar{\theta}_n^{(n)}}\right), & x < \alpha_n^{(n)}. \end{cases} \tag{2·29}$$

Then, from (2·26) to (2·28) and (1·45), (1·48), we see that $f_{ij}^{(n)}(t)$ is given by

$$f_{ij}^{(n)}(t) = -l_{i-1} m_j \int x \phi_{i-1}(x)\, \phi_{j-1}(x)\, e^{tx}\, d\bar{\rho}^{(n)}(x) \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n), \tag{2·30a}$$

$$f_{0j}^{(n)}(t) = \delta_{0j} \quad (0 \leqslant j \leqslant n), \tag{2·30b}$$

$$f_{i0}^{(n)}(t) = \zeta_i^{(n)} - l_{i-1} \int \phi_{i-1}(x)\, e^{tx}\, d\bar{\rho}^{(n)}(x) \quad (1 \leqslant i \leqslant n). \tag{2·30c}$$

Similarly, we define
$$\bar{\eta}_r^{(n)} = \eta_r^{(n)} / |\beta_r^{(n)}| \tag{2·31}$$

and $\bar{\sigma}^{(n)}(x)$ to have discontinuities $1/\bar{\eta}_r^{(n)}$ at $x = \beta_r^{(n)}$ for $r = 1, 2, \ldots, n$:

$$\bar{\sigma}^{(n)}(x) = \begin{cases} 0, & x \geqslant \beta_1^{(n)}; \\ -\left(\dfrac{1}{\bar{\eta}_1^{(n)}} + \ldots + \dfrac{1}{\bar{\eta}_r^{(n)}}\right), & \beta_{r+1}^{(n)} \leqslant x < \beta_r^{(n)} \quad (r = 1, \ldots, n-1); \\ -\left(\dfrac{1}{\bar{\eta}_1^{(n)}} + \ldots + \dfrac{1}{\bar{\eta}_n^{(n)}}\right), & x < \beta_n^{(n)}. \end{cases} \tag{2·32}$$

Then
$$g_{ij}^{(n)}(t) = -l_{i-1} m_j \int x \bar{\phi}_{i-1}(x)\, \bar{\phi}_{j-1}(x)\, e^{tx}\, d\bar{\sigma}^{(n)}(x) \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n), \tag{2·33a}$$
$$g_{0j}^{(n)}(t) = \delta_{0j} \quad (0 \leqslant j \leqslant n), \tag{2·33b}$$
$$g_{i0}^{(n)}(t) = 1 - l_{i-1} \int \bar{\phi}_{i-1}(x)\, e^{tx}\, d\bar{\sigma}^{(n)}(x) \quad (1 \leqslant i \leqslant n). \tag{2·33c}$$

In particular, since $f_{i0}^{(n)}(0) = g_{i0}^{(n)}(0) = 0$ for $1 \leqslant i \leqslant n$, (2·30c) and (2·33c) imply that
$$\int \bar{\phi}_{i-1}(x)\, d\bar{\rho}^{(n)}(x) = \zeta_i^{(n)}/l_{i-1}, \tag{2·34}$$
$$\int \bar{\phi}_{i-1}(x)\, d\bar{\sigma}^{(n)}(x) = 1/l_{i-1}. \tag{2·35}$$

Since x^m $(m \geqslant 0)$ is a linear combination of $\phi_0(x), \ldots, \phi_m(x)$, and $\zeta_i^{(n)} \leqslant 1$, we can deduce as in §4 that
$$\left.\begin{aligned} \int |x|^m\, d\bar{\rho}^{(n)}(x) \leqslant K_m \\ \int |x|^m\, d\bar{\sigma}^{(n)}(x) \leqslant K_m \end{aligned}\right\} \quad (m = 0, 1, 2, \ldots;\ n > m). \tag{2·36}$$

This enables us to repeat the arguments of §4, selecting subsequences $\bar{\rho}^{(n_k)}(x)$, $\bar{\sigma}^{(n_k)}(x)$ of $\bar{\rho}^{(n)}(x)$, $\bar{\sigma}^{(n)}(x)$, with limit functions $\bar{\rho}(x)$, $\bar{\sigma}(x)$. We need only observe in addition that, by (1·41), $\zeta_i^{(n)}$ increases and tends to a limit ζ_i as $n \to \infty$, where
$$\zeta_0 = 1, \quad \zeta_i = 1 - \left(\sum_{\nu=0}^{i-1} w_\nu^{-1}\right) \Big/ \left(\sum_{\nu=0}^{\infty} w_\nu^{-1}\right) \quad (i \geqslant 1); \tag{2·37}$$

alternatively
$$\zeta_i = 1 \quad (i \geqslant 0) \quad \text{if} \quad \sum_0^\infty w_\nu^{-1} = \infty, \tag{2·38}$$
$$\zeta_i = \left(\sum_{\nu=i}^{\infty} w_\nu^{-1}\right) \Big/ \left(\sum_{\nu=0}^{\infty} w_\nu^{-1}\right) \quad \text{if} \quad \sum_0^\infty w_\nu^{-1} < \infty. \tag{2·39}$$

Hence $f_{ij}^{(n_k)}(t)$, $g_{ij}^{(n_k)}(t)$ tend to limits $f_{ij}(t)$, $g_{ij}(t)$, where
$$f_{ij}(t) = -l_{i-1} m_j \int x \bar{\phi}_{i-1}(x)\, \bar{\phi}_{j-1}(x)\, e^{tx}\, d\bar{\rho}(x) \quad (i \geqslant 1, j \geqslant 1), \tag{2·40a}$$
$$f_{0j}(t) = \delta_{0j} \quad (j \geqslant 0), \tag{2·40b}$$
$$f_{i0}(t) = \zeta_i - l_{i-1} \int \bar{\phi}_{i-1}(x)\, e^{tx}\, d\bar{\rho}(x) \quad (i \geqslant 1); \tag{2·40c}$$
$$g_{ij}(t) = -l_{i-1} m_j \int x \bar{\phi}_{i-1}(x)\, \bar{\phi}_{j-1}(x)\, e^{tx}\, d\bar{\sigma}(x), \tag{2·41a}$$
$$g_{0j}(t) = \delta_{0j} \quad (j \geqslant 0), \tag{2·41b}$$
$$g_{i0}(t) = 1 - l_{i-1} \int \bar{\phi}_{i-1}(x)\, e^{tx}\, d\bar{\sigma}(x) \quad (i \geqslant 1). \tag{2·41c}$$

Both $f_{ij}(t)$ and $g_{ij}(t)$ are solutions of the system of equations (0·14), and have the properties
$$\left.\begin{aligned} &0 \leqslant f_{ij}(t) \leqslant g_{ij}(t); \\ &\sum_{j=0}^{\infty} f_{ij}(t) \leqslant 1, \quad \sum_{j=0}^{\infty} g_{ij}(t) \leqslant 1; \end{aligned}\right\} \quad (t \geqslant 0)$$
$$f_{ij}(0) = g_{ij}(0) = \delta_{ij}.$$

As in §4, the sequences $f_{ij}^{(n)}(t)$ and $\bar{\rho}^{(n)}(x)$ converge as they stand and no selection process is needed to obtain $f_{ij}(t)$ and $\bar{\rho}(x)$. Also (2·24) is again valid.

6. *Asymptotic values*

The results of §§4 and 5 enable us to deduce the behaviour of $f_{ij}(t)$ and $g_{ij}(t)$ as $t \to \infty$. We recall the well-known fact that, if $P(x)$ is a polynomial, $\rho(x)$ non-decreasing, and

$$\int_{-\infty}^{0} |P(x)|\, d\rho(x) < \infty,$$

then

$$\lim_{t\to\infty} \int_{-\infty}^{0} P(x)\, e^{tx}\, d\rho(x) = \rho_0 P(0), \tag{2·42}$$

where $\rho_0 = \rho(+0) - \rho(-0)$ is the discontinuity of $\rho(x)$ at $x = 0$.† Since the integrals in the expressions for $f_{ij}(t)$, $g_{ij}(t)$ (cf. (2·14), (2·18), (2·40 *a*, *b*, *c*), and (2·41 *a*, *b*, *c*)) are all of the form occurring in (2·42), it follows that as $t \to \infty$, $f_{ij}(t)$ and $g_{ij}(t)$ tend to limits which we shall denote by

$$f_{ij}(\infty) = \lim_{t\to\infty} f_{ij}(t), \quad g_{ij}(\infty) = \lim_{t\to\infty} g_{ij}(t). \tag{2·43}$$

Before deriving explicit formulae for these quantities from (2·42), we shall show that some information can be obtained merely from the fact that $f_{ij}(\infty)$ and $g_{ij}(\infty)$ exist. First, it follows easily from (2·17), (2·21) and (2·23) (which are valid also when $\lambda_0 = 0$) that

$$0 \leqslant f_{ij}(\infty) \leqslant 1, \quad 0 \leqslant g_{ij}(\infty) \leqslant 1; \tag{2·44}$$

$$\sum_{j=0}^{\infty} f_{ij}(\infty) \leqslant 1, \quad \sum_{j=0}^{\infty} g_{ij}(\infty) \leqslant 1; \tag{2·45}$$

$$f_{ij}(\infty) \leqslant g_{ij}(\infty). \tag{2·46}$$

Secondly, since $\lim_{t\to\infty} f_{ij}(t)$ exists (for each i and j), equation (2·16) shows that

$$\lim_{t\to\infty} f'_{ij}(t)$$

also exists, and must therefore vanish. Hence

$$\lambda_{j-1} f_{i,j-1}(\infty) - (\lambda_j + \mu_j) f_{ij}(\infty) + \mu_{j+1} f_{i,j+1}(\infty) = 0, \tag{2·47}$$

or more concisely (with an obvious notation)

$$F(\infty)\, A = 0. \tag{2·48}$$

This determines each row $f_{ij}(\infty)$ (i fixed) of $F(\infty)$ except for a factor c_i; we must have

$$f_{ij}(\infty) = c_i \frac{m_j}{l_{j-1}} \quad \text{if} \quad \lambda_0 > 0, \tag{2·49}$$

$$f_{ij}(\infty) = c_i \delta_{0j} \quad \text{if} \quad \lambda_0 = 0. \tag{2·50}$$

Similarly for $g_{ij}(\infty)$, c_i being replaced by c'_i, where $c'_i \geqslant c_i$ from (2·46).

If $\lambda_0 > 0$ and $\sum_0^{\infty} \frac{m_j}{l_{j-1}}$ diverges, it follows from (2·45) and (2·49) that $c_i = 0$, and similarly $c'_i = 0$. Hence

$$f_{ij}(\infty) = g_{ij}(\infty) = 0 \quad \text{if} \quad \lambda_0 > 0, \quad \sum_{j=0}^{\infty} \frac{m_j}{l_{j-1}} = \infty. \tag{2·51}$$

† We can find no precise reference; see Widder (1941, p. 181) for the type of argument required.

If $\lambda_0 = 0$, we have only to determine $f_{i0}(\infty)$ and $g_{i0}(\infty)$, since $f_{ij}(\infty) = g_{ij}(\infty) = 0$ for $j>0$ (by (2·50)). Now (2·16), for $j=0$, reads

$$f'_{i0}(t) = \mu_1 f_{i1}(t),$$

so that $f'_{i0}(t) \geq 0$ and $f_{i0}(t)$ increases steadily to its limit $f_{i0}(\infty)$. Moreover, by theorem 4,

$$f_{i0}(t) = \lim_{n\to\infty} f_{i0}^{(n)}(t) \geq f_{i0}^{(n)}(t) \quad (n \geq i). \tag{2·52}$$

Finally, from (1·48), we see that

$$f_{i0}^{(n)}(\infty) = d_0^{(n)}(i, 0) = \zeta_i^{(n)}.$$

On combining the foregoing facts, it follows that

$$f_{i0}(\infty) = \zeta_i \quad (\lambda_0 = 0). \tag{2·53}$$

For if t is fixed, then

$$f_{i0}(t) = \lim_{n\to\infty} f_{i0}^{(n)}(t) \leq \lim_{n\to\infty} f_{i0}^{(n)}(\infty) = \lim_{n\to\infty} \zeta_i^{(n)} = \zeta_i,$$

and on letting $t\to\infty$, $f_{i0}(\infty) \leq \zeta_i$. On the other hand, from (2·52),

$$f_{i0}(\infty) \geq f_{i0}^{(n)}(\infty) = \zeta_i^{(n)} \quad (n \geq i),$$

and on letting $n\to\infty$, we obtain $f_{i0}(\infty) \geq \zeta_i$. In particular, using (2·38) and the fact that

$$f_{i0}(\infty) \leq g_{i0}(\infty) \leq 1,$$

(2·53) implies that

$$f_{i0}(\infty) = g_{i0}(\infty) = 1 \quad \text{if} \quad \lambda_0 = 0 \quad \text{and} \quad \sum_{\nu=0}^{\infty} w_\nu^{-1} = \infty.\dagger \tag{2·54}$$

These results have been derived without using the explicit expressions for $f_{ij}(\infty)$, $g_{ij}(\infty)$ obtainable from (2·42). If we do use them, we can obtain further results (and confirm what has been proved so far in (2·49), (2·50) and (2·53)). If $\lambda_0>0$, we use (2·14) and (2·18). Denoting the discontinuities of $\rho(x)$, $\sigma(x)$ at $x=0$ by ρ_0, σ_0, we find from (2·42) that

$$f_{ij}(\infty) = \rho_0 l_{i-1} m_j \phi_{i-1}(0)\, \phi_{j-1}(0),$$

$$g_{ij}(\infty) = \sigma_0 l_{i-1} m_j \phi_{i-1}(0)\, \phi_{j-1}(0).$$

Since, by (1·24), $\phi_{k-1}(0) = \lambda_0\lambda_1\ldots\lambda_{k-1} = \lambda_0/l_{k-1}$, these may be written as

$$f_{ij}(\infty) = \rho_0 \lambda_0^2 \frac{m_j}{l_{j-1}}, \tag{2·55}$$

$$g_{ij}(\infty) = \sigma_0 \lambda_0^2 \frac{m_j}{l_{j-1}}. \tag{2·56}$$

This confirms (2·49) (and the companion for $g_{ij}(\infty)$); it shows furthermore, that the factors c_i (and c'_i) are independent of i. Since i specifies the initial values we conclude that, when $\lambda_0>0$, the system has the ergodic property, in the sense that the asymptotic distribution is independent of the initial distribution. From (2·46) we have

$$\sigma_0 \geq \rho_0, \tag{2·57}$$

† In fact, by D, theorem 6 (iv), $f_{ij}(t) \equiv g_{ij}(t)$ in this case.

while from (2·51) we know that $\sigma_0 = \rho_0 = 0$ if $\sum_0^\infty \frac{m_j}{l_{j-1}} = \infty$. However, σ_0 can also be calculated if $\sum_0^\infty \frac{m_j}{l_{j-1}} < \infty$. For $\sigma^{(n)}(x)$ has a discontinuity $1/\eta_0^{(n)}$ at $x = \beta_0^{(n)} = 0$, where (cf. (1·71))

$$\eta_0^{(n)} = \lambda_0 + \lambda_0^2 \sum_{\nu=1}^{n} \frac{w_{\nu-1}}{\mu_\nu} = \lambda_0^2 \sum_{\nu=0}^{n} \frac{m_\nu}{l_{\nu-1}} \leqslant \lambda_0^2 \sum_{\nu=0}^{\infty} \frac{m_\nu}{l_{\nu-1}}.$$

Hence, for any $x < 0$,
$$\sigma^{(n)}(0) - \sigma^{(n)}(x) \geqslant (\eta_0^{(n)})^{-1} \geqslant \left(\lambda_0^2 \sum_0^\infty \frac{m_\nu}{l_{\nu-1}}\right)^{-1},$$

and therefore
$$\sigma_0 \geqslant \sigma(0) - \sigma(x) \geqslant \left(\lambda_0^2 \sum_0^\infty \frac{m_\nu}{l_{\nu-1}}\right)^{-1}.$$

On the other hand, from (2·56) and (2·21),

$$\sigma_0 \lambda_0^2 \sum_0^\infty \frac{m_\nu}{l_{\nu-1}} \leqslant 1.$$

Hence
$$\sigma_0 = \left(\lambda_0^2 \sum_0^\infty \frac{m_\nu}{l_{\nu-1}}\right)^{-1},$$

and we have shown that by (1·31)

$$g_{ij}(\infty) = \frac{m_j}{l_{j-1}} \left(\sum_0^\infty \frac{m_\nu}{l_{\nu-1}}\right)^{-1} \quad \text{if} \quad \lambda_0 > 0, \quad \sum_0^\infty \frac{m_\nu}{l_{\nu-1}} < \infty. \tag{2·58}$$

The calculation of ρ_0 by a similar method would be much more difficult, because $\rho^{(n)}(x)$ has no discontinuity at $x = 0$ (since $\alpha_0^{(n)} < 0$), and we have been unable to devise any method of deciding whether $\alpha_0^{(n)} \to 0$ as $n \to \infty$, and if so whether $1/\theta_0^{(n)}$ has a positive lower bound.

If $\lambda_0 = 0$, we use (2·40 *a*, *b*, *c*) and (2·41 *a*, *b*, *c*), and denote the discontinuities of $\bar{\rho}(x)$, $\bar{\sigma}(x)$ at $x = 0$ by $\bar{\rho}_0$, $\bar{\sigma}_0$. From (2·42), we find that

$$f_{ij}(\infty) = g_{ij}(\infty) = 0 \quad \text{if} \quad j > 0,$$

confirming (2·50), and that

$$f_{i0}(\infty) = \zeta_i - \bar{\rho}_0 \sum_{\nu=0}^{i-1} \frac{l_\nu}{m_\nu} = \zeta_i - \bar{\rho}_0 \sum_0^{i-1} w_\nu^{-1}, \tag{2·59}$$

$$g_{i0}(\infty) = 1 - \bar{\sigma}_0 \sum_{\nu=0}^{i-1} \frac{l_\nu}{m_\nu} = 1 - \bar{\sigma}_0 \sum_0^{i-1} w_\nu^{-1}. \tag{2·60}$$

Closer consideration of (2·59) shows that $\bar{\rho}_0 = 0$ and hence confirms (2·53); for $\bar{\rho}_0 > 0$ would contradict $f_{i0}(\infty) \geqslant 0$ for large i, because either $\sum_0^\infty w_\nu^{-1} = \infty$ and $\zeta_i = 1$, or $\sum_0^\infty w_\nu^{-1} < \infty$ and $\zeta_i \to 0$ (cf. (2·38) and (2·39)). The same argument shows that $\bar{\sigma}_0 = 0$ if $\sum_0^\infty w_\nu^{-1} = \infty$, but we cannot in general evaluate $\bar{\sigma}_0$ when $\sum_0^\infty w_\nu^{-1} < \infty$. Hence (2·53) and (2·54) are confirmed, and

$$\left.\begin{aligned} &f_{i0}(\infty) = g_{i0}(\infty) = 1 \quad \text{if} \quad \sum_0^\infty w_\nu^{-1} = \infty, \\ &f_{i0}(\infty) = \zeta_i, \quad g_{i0}(\infty) = 1 - \bar{\sigma}_0 \sum_0^{i-1} w_\nu^{-1} \quad \text{if} \quad \sum_0^\infty w_\nu^{-1} < \infty. \end{aligned}\right\} \tag{2·61}$$

As the right-hand sides of (2·59) and (2·60) in general depend on i, the system need not be ergodic when $\lambda_0 = 0$. The result (2·61) sheds some light on the question whether $f_{ij}(t)$ and

$g_{ij}(t)$ are distinct. We see that, if $\sum_0^\infty w_\nu^{-1} < \infty$ and $\bar{\sigma}_0 = 0$, then $f_{i0}(\infty) \neq g_{i0}(\infty)$ and *a fortiori* $f_{i0}(t) \not\equiv g_{i0}(t)$ (for $i > 0$). A sufficient condition for $\bar{\sigma}_0 = 0$ is that $\bar{\sigma}(x)$ should be constant in some interval $(-\delta, 0)$, and a sufficient condition for this is that $|\beta_1^{(n)}| \geq \delta > 0$ for all n. We shall show in chapter III that, for a suitable choice of λ_j and μ_j, this condition can be satisfied simultaneously with $\sum_0^\infty w_\nu^{-1} < \infty$, and hence that $f_{ij}(t)$ and $g_{ij}(t)$ may be distinct solutions of (0·14).

7. *The nature of the spectrum*

It is natural to inquire a little more closely into the nature of the spectral functions $\rho(x)$, $\sigma(x)$, $\bar{\rho}(x)$ and $\bar{\sigma}(x)$ which have appeared in this chapter. We recall from (1·34) that for fixed $r \geq 0$

$$\alpha_r^{(n)} \leq \alpha_{r+1}^{(n)} \leq 0 \quad (n = r, r+1, \ldots).$$

It follows that $\alpha_r^{(n)}$ tends to a limit as $n \to \infty$, say

$$\alpha_r^{(n)} \to \alpha_r \quad \text{as} \quad n \to \infty, \tag{2·62}$$

where clearly

$$0 \geq \alpha_0 \geq \alpha_1 \geq \alpha_2 \geq \ldots. \tag{2·63}$$

Similarly we infer from (1·65) that

$$\beta_r^{(n)} \to \beta_r \quad \text{as} \quad n \to \infty, \tag{2·64}$$

where

$$0 = \beta_0 \geq \beta_1 \geq \beta_2 \geq \ldots. \tag{2·65}$$

Finally, by (1·87),

$$\alpha_r \leq \beta_r \quad (r = 0, 1, \ldots). \tag{2·66}$$

(Note that $\alpha_0 = 0$ when $\lambda_0 = 0$.)

If no restrictions are put on λ_n and μ_n (other than $\lambda_0 \geq 0$, $\lambda_n > 0$, $\mu_n > 0$ for $n \geq 1$), nothing can be said about the α_r (and β_r), and they cannot be easily related to the functions $\rho(x), \ldots, \bar{\sigma}(x)$. It may, for instance, happen that all the α_r (or β_r) coincide, and that the spectral functions $\rho(x), \ldots$ are continuous (see chapter IV, §§13, 14). Suppose, however, that the α_r are all distinct, i.e. that

$$0 \geq \alpha_0 > \alpha_1 > \alpha_2 > \ldots, \tag{2·67}$$

and have no finite limit point, so that $\alpha_r \to -\infty$ as $r \to \infty$. Then $\rho(x)$ is a step-function with discontinuities at $x = \alpha_0, \alpha_1, \ldots$. For, considering first the interval between α_0 and α_1, let $0 < \epsilon < \alpha_0 - \alpha_1$. Then, for $n > n_0(\epsilon)$,

$$\alpha_0 > \alpha_0^{(n)} > \alpha_0 - \epsilon > \alpha_1 > \alpha_1^{(n)}.$$

Since $\rho^{(n)}(x) = 0$ for $x \geq \alpha_0^{(n)}$, $= -\dfrac{1}{\theta_0^{(n)}}$ for $\alpha_0^{(n)} > x \geq \alpha_1^{(n)}$, and $\rho^{(n)}(x) \to \rho(x)$ at points of continuity of $\rho(x)$, it follows easily that

$$\rho(x) = \begin{cases} 0 & \text{for} \quad x \geq \alpha_0, \\ -\dfrac{1}{\theta_0} & \text{for} \quad \alpha_0 > x \geq \alpha_1, \end{cases}$$

where θ_0 is given by

$$\lim_{n\to\infty} \frac{1}{\theta_0^{(n)}} = \frac{1}{\theta_0}.$$

Thus $\rho(x)$ has a discontinuity $1/\theta_0$ at $x = \alpha_0$, and is constant between α_0 and α_1 (though it may happen that θ_0 is infinite and the discontinuity at α_0 is zero). Similarly, we may show that $\rho(x)$ has a discontinuity

$$\frac{1}{\theta_1} = \lim_{n\to\infty} \frac{1}{\theta_1^{(n)}}$$

at α_1, and is constant between α_1 and α_2; and so on. Similar descriptions (with slight modifications) apply if (2·67) is relaxed by allowing some of the α_r to coincide, but only with finite multiplicity, say

$$0 \geqslant \alpha_0 = \alpha_1 = \ldots = \alpha_{s_1} > \alpha_{s_1+1} = \ldots = \alpha_{s_2} > \alpha_{s_2+1} = \ldots; \tag{2·68}$$

we may say in this case that the spectrum is discrete. A sufficient condition for this to occur is that, for some $\gamma > 0$,

$$\sum_{r=1}^{\infty} |\alpha_r|^{-\gamma} < \infty, \tag{2·69}$$

or equivalently that

$$\sum_{r=1}^{n} |\alpha_r^{(n)}|^{-\gamma} \leqslant C < \infty \quad (n = 1, 2, \ldots). \tag{2·70}$$

This not only implies that (2·68) holds, but also that $\alpha_1 < 0$; similarly

$$\sum_{r=1}^{\infty} |\beta_r|^{-\gamma} < \infty \tag{2·71}$$

implies (2·68) with β_r replaced by α_r, and also implies that $\beta_1 < 0$. (Note, from (2·66), that (2·71) implies (2·69).) In particular, when $\lambda_0 = 0$ and (2·71) holds, $\bar{\sigma}(x) = 0$ for $x \geqslant \beta_1$ and $\beta_1 < 0$, so that $\bar{\sigma}_0 = 0$ and (cf. (2·60)) $g_{i0}(\infty) = 1$.

It will be shown in the next chapter that (2·71) or (2·69) holds in some cases when restrictions are imposed on λ_j and μ_j. Moreover, it will be shown that $\theta_0, \theta_1, \ldots$ can be expressed in a form analogous to $\theta_0^{(n)}, \theta_1^{(n)}, \ldots$ (cf. (1·43)), i.e.

$$\theta_r = \sum_{\nu=0}^{\infty} l_{\nu-1} m_\nu \{\phi_{\nu-1}(\alpha_r)\}^2.$$

The Stieltjes integrals occurring in the present chapter reduce to infinite series when the spectrum is discrete (so that $\rho(x), \ldots$ reduce to step-functions). The resulting expressions for $f_{ij}(t)$, $g_{ij}(t)$ will be derived independently in chapter III, without using the general results of the present chapter.

Chapter III. Analytical processes

8. *Definition*

The quantities α_r, β_r $(r = 0, 1, 2, \ldots)$ have been defined in chapter II, § 7. For convenience we note here the main facts about these numbers:

$$\alpha_r = \lim_{n\to\infty} \alpha_r^{(n)}, \tag{3·1}$$

$$\beta_r = \lim_{n\to\infty} \beta_r^{(n)}, \tag{3·2}$$

$$|\alpha_r^{(n)}| \geqslant |\alpha_r| \quad (n = r, r+1, \ldots), \tag{3·3}$$

$$|\beta_r^{(n)}| \geqslant |\beta_r| \quad (n = r, r+1, \ldots), \tag{3·4}$$

$$\left.\begin{array}{l} 0 \geqslant \alpha_0 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \ldots, \\ 0 = \beta_0 \geqslant \beta_1 \geqslant \beta_2 \geqslant \ldots, \end{array}\right\} \tag{3·5}$$

$$|\beta_r| \geqslant |\alpha_r| \quad (r = 0, 1, 2, \ldots). \tag{3·6}$$

We recall that $\alpha_0 = 0$ if $\lambda_0 = 0$.

At the end of the preceding chapter certain deductions were made from the assumption that A has a discrete spectrum. We do not know the necessary and sufficient conditions for A to have this property, and we shall therefore be content to use a simple set of sufficient conditions. Broadly speaking, we shall assume that λ_n and μ_n grow like the same power of n and that their ratio tends to a positive limit as n tends to ∞. More precisely, we shall, throughout this chapter, consider only those processes for which

$$\frac{\lambda_{n+1}}{\lambda_n} = 1+\frac{a}{n}+O\left(\frac{1}{n^2}\right), \tag{3·7}$$

$$\frac{\lambda_n}{\mu_n} = c\left(1+\frac{b}{n}+O\left(\frac{1}{n^2}\right)\right) \quad (c>0), \tag{3·8}$$

where a, b and c are constants. A process of this kind will be termed *analytical*, and a, b, c its *parameters*.

We have decided to devote a considerable amount of space to a discussion of these special processes because they afford concrete examples in which the structure of the final solution is comparatively simple. In this chapter we shall also present cases in which $F(t) \neq G(t)$, thus demonstrating the possibility of non-uniqueness of the solution.

We observe that (3·7) implies that

$$\lambda_n = \lambda n^a\left(1+O\left(\frac{1}{n}\right)\right), \tag{3·9}$$

where λ is a suitable constant. For

$$\ln\lambda_n - \ln\lambda_1 = \sum_{j=1}^{n-1}\ln\frac{\lambda_{j+1}}{\lambda_j} = \sum_{j=1}^{n-1}\ln\left(1+\frac{a}{j}+O\left(\frac{1}{j^2}\right)\right)$$

$$= a\sum_{j=1}^{n-1}\frac{1}{j}+\Sigma\, O\left(\frac{1}{j^2}\right),$$

and hence
$$\ln\lambda_n = \text{const.}+a\ln n+O\left(\frac{1}{n}\right). \tag{3·10}$$

Evidently, (3·10) implies (3·9). By (3·8)

$$\mu_n = \frac{\lambda}{c}n^a\left(1+O\left(\frac{1}{n}\right)\right). \tag{3·11}$$

Similarly, on applying (3·8) to (1·30) we find that

$$w_n = c^n w n^b\left(1+O\left(\frac{1}{n}\right)\right). \tag{3·12}$$

Under suitable conditions (to be mentioned later) on the constants a, b and c we shall prove that one or both of the infinite series

$$\sum_{r=1}^{\infty}|\alpha_r|^{-2}, \quad \sum_{r=1}^{\infty}|\beta_r|^{-2} \tag{3·13}$$

converge, which implies the discreteness of the spectrum (see chapter II, §7). Finally, after showing that, for fixed r, $\theta_r^{(n)}$ and $\eta_r^{(n)}$ (see (1·42) and (1·70)) tend to finite limits, as $n\to\infty$, we shall derive solutions of the problem by letting $n\to\infty$ in the formulae for $f_{ij}^{(n)}(t)$ and $g_{ij}^{(n)}(t)$. This method of constructing solutions, which of course applies only to the restricted class of process under discussion, is independent of the general theory developed in chapter II.

9. *Sufficient conditions for a discrete spectrum*

By (3·6) the convergence of the second series in (3·13) implies that of the first. We shall therefore begin by considering in more detail the eigenvalues of the modified sections $B^{(n)}$. Writing the characteristic polynomial in the form

$$\psi_n(\xi) = \mu_1\mu_2\ldots\mu_n(a_n\xi+b_n\xi^2+c_n\xi^3+\ldots) \quad (n\geqslant 1), \tag{3·14}$$

we see that the recurrence relation (1·55) is equivalent to

$$\mu_n(\Delta a_n\xi+\Delta b_n\xi^2+\Delta c_n\xi^3+\ldots)-\lambda_{n-1}(\Delta a_{n-1}\xi+\Delta b_{n-1}\xi^2+\Delta c_{n-1}\xi^3+\ldots) = \xi(a_{n-1}\xi+b_{n-1}\xi^2+c_{n-1}\xi^3+\ldots), \tag{3·15}$$

where, generally, $\Delta u_n = u_n - u_{n-1}$.

Comparing coefficients in (3·15), we obtain, for $n\geqslant 1$,

$$\mu_n\Delta a_n - \lambda_{n-1}\Delta a_{n-1} = 0, \tag{3·16}$$

$$\mu_n\Delta b_n - \lambda_{n-1}\Delta b_{n-1} = b_{n-1}, \tag{3·17}$$

$$\mu_n\Delta c_n - \lambda_{n-1}\Delta c_{n-1} = c_{n-1}, \tag{3·18}$$

the initial values being given by

$$a_{-1} = b_{-1} = c_{-1} = 0, \quad a_0 = 1, \quad b_0 = c_0 = 0. \tag{3·19}$$

Equations (3·16) to (3·19) lead to the following explicit expressions for a_n, Δb_n and Δc_n:

$$a_n = 1+\lambda_0\sum_{r=1}^{n}\frac{w_{r-1}}{\mu_r} \quad (n\geqslant 1), \tag{3·20}$$

$$\Delta b_n = \frac{1}{\mu_n}w_{n-1}\sum_{r=0}^{n-1}\frac{a_r}{w_r} \quad (n\geqslant 1), \tag{3·21}$$

$$\Delta c_n = \frac{1}{\mu_n}w_{n-1}\sum_{r=0}^{n-1}\frac{b_r}{w_r} \quad (n\geqslant 1). \tag{3·22}$$

In order to prove the convergence of the second series (3·13) we study the corresponding finite sum

$$t_n = \sum_{r=1}^{n}|\beta_r^{(n)}|^{-2}.$$

Since the $\beta_r^{(n)}$ $(r = 1, 2, \ldots, n)$ are the roots of

$$a_n+b_n\xi+c_n\xi^2+\ldots = 0,$$

we have

$$t_n = \left(\frac{b_n}{a_n}\right)^2 - 2\left(\frac{c_n}{a_n}\right). \tag{3·23}$$

Suppose we can show that, for a given set of λ's and μ's,

$$t_n \leqslant h \tag{3·24}$$

for all n. Then, if N is an arbitrary integer

$$\sum_{r=1}^{N}|\beta_r^{(n)}|^{-2} \leqslant h \quad (n\geqslant N),$$

and, as $n \to \infty$,
$$\sum_{r=1}^{N} |\beta_r|^{-2} \leqslant h,$$
which implies the convergence of the series (3·13) and, in fact, shows that
$$\sum_{r=1}^{\infty} |\beta_r|^{-2} \leqslant h.$$
We shall therefore endeavour to establish (3·24).

The asymptotic evaluation of t_n is based on two lemmas:

LEMMA 2. *Let $u_1, u_2, u_3, \ldots$ be an infinite sequence of positive numbers such that*
$$\frac{u_{r-1}}{u_r} = q + O\left(\frac{1}{r}\right), \tag{3·25}$$
where q is a fixed positive number less than unity. Then
$$\sum_{r=1}^{n} u_r = u_n\left\{\frac{1}{1-q} + O\left(\frac{1}{n}\right)\right\}.$$

Proof. Since
$$\sum_{r=1}^{n} u_r = u_n \sum_{r=0}^{n-1} \frac{u_{n-r}}{u_n},$$
we have to show that
$$\sum_{r=0}^{n-1} \frac{u_{n-r}}{u_n} = \frac{1}{1-q} + O\left(\frac{1}{n}\right). \tag{3·26}$$
The hypothesis (3·25) can be written in the form
$$\frac{u_{r-1}}{u_r} = q(1+v_r),$$
where
$$|v_r| \leqslant C/r, \tag{3·27}$$
C, and later, $C', C'', \ldots$ being suitable constants. Then
$$\frac{u_{n-r}}{u_n} = q^r(1+v_{n-r+1})(1+v_{n-r+2})\ldots(1+v_n) = q^r(1+z_r), \tag{3·28}$$
say. It is easily seen that
$$1+|z_r| \leqslant \prod_{j=1}^{r}(1+|v_{n-r+j}|) \leqslant \exp\left(\sum_{j=1}^{r}|v_{n-r+j}|\right).$$
In virtue of (3·27) we can estimate the last sum in two different ways, namely,
$$\sum_{j=1}^{r}|v_{n-r+j}| \begin{cases} \leqslant Cr/(n-r+1), \\ \leqslant C' \ln n, \end{cases}$$
the former being more effective for small values of r and the latter for large. More precisely, assuming that n is so great that
$$k = \left[-\frac{C'+1}{\ln q}\ln n\right] \leqslant \tfrac{1}{2}n,$$
we shall use the first inequality for $r \leqslant k$ and the second for $r > k$.

By (3·28)
$$\sum_{r=0}^{n-1} \frac{u_{n-r}}{u_n} = \sum_{r=0}^{n-1} q^r + \sum_{r=0}^{n-1} z_r q^r. \tag{3·29}$$
Write
$$\sum_{r=0}^{n-1} z_r q^r = \sum_{r=0}^{k} z_r q^r + \sum_{r=k+1}^{n-1} z_r q^r. \tag{3·30}$$

For $r \leqslant k$, we have $$1+|z_r| \leqslant \exp\left(\frac{Cr}{n-r+1}\right) \leqslant \exp\left(\frac{2Cr}{n+2}\right) < \exp\frac{2Cr}{n},$$

and hence $$|z_r| < \frac{C''r}{n} \quad (r \leqslant k).$$

On the other hand, for $r>k$, $\quad 1+|z_r| \leqslant \exp(C'\ln n) = n^{C'}$,

$$|z_r|\, q^{k+1} < n^{C'} q^{k+1} < \frac{1}{n},$$

by definition of k. Substituting in (3·30) we find that

$$\left|\sum_{r=0}^{n-1} z_r q^r\right| < \frac{C''}{n}\sum_{r=0}^{k} rq^r + \sum_{r=k+1}^{n-1} |z_r|\, q^{k+1} q^{r-(k+1)} < \frac{C''}{n}\sum_{r=0}^{k} rq^r + \frac{1}{n}\sum_{r=k+1}^{n-1} q^{r-(k+1)} = O\left(\frac{1}{n}\right).$$

Since $$\sum_{r=0}^{n-1} q^r = \frac{1}{1-q} + O(q^n) = \frac{1}{1-q} + o\left(\frac{1}{n}\right),$$

(3·29) can be reduced to (3·26), and the lemma follows.

LEMMA 3. *For any real value of* α

$$\sum_{r=1}^{n} r^{\alpha} = O(n^{\alpha+1}\ln n + 1). \tag{3·31}$$

Proof. The assertion (3·31) is either equivalent to, or weaker than, the well-known results obtained by comparing the sum with an integral.

We now return to the main problem of this section.

THEOREM 5. *In an analytical process with parameters a, b, c we have*

$$\sum_{r=1}^{\infty} |\beta_r|^{-2} < \infty \quad \textit{and hence} \quad \sum_{r=1}^{\infty} |\alpha_r|^{-2} < \infty,$$

if either $$\text{(i)} \quad 0<c<1, \quad a>\tfrac{1}{2}, \tag{3·32}$$

or $$\text{(ii)} \quad c=1, \quad a>2, \quad a-b>1. \tag{3·33}$$

Proof. As we have already remarked, it is sufficient to establish (3·24).

Suppose first that the conditions (i) are satisfied. Since, by (3·11) and (3·12),

$$w_{r-1}/\mu_r = O(c^r r^{b-a}),$$

we have $$\sum_{r=1}^{\infty} \frac{w_{r-1}}{\mu_r} < \infty.$$

Hence we see from (3·20) that a_n tends to a finite limit, say

$$\lim_{n\to\infty} a_n = a_\infty,$$

where $$1 \leqslant a_n \leqslant a_\infty < \infty.$$

For future reference we note that

$$\frac{\Delta a_r}{a_r} = \frac{\lambda_0 w_{r-1}}{a_r \mu_r} = O(c^r r^{b-a}) = o\left(\frac{1}{r}\right).$$

Again, if d is any number satisfying $\quad c<d<1$,

we can write $$a_\infty - a_{n-1} = \lambda_0 \sum_{r=n}^{\infty} \frac{w_{r-1}}{\mu_r} = O(d^n) = o\left(\frac{1}{n}\right), \tag{3·34}$$

because $$c^r r^{b-a} = d^r \left(\frac{d}{c}\right)^r r^{b-a} = d^r O(1).$$

In order to obtain an estimate for the sum in (3·21) put

$$u_r = a_r/w_r.$$

Then
$$\frac{u_{r-1}}{u_r} = \frac{a_{r-1}}{a_r}\frac{w_r}{w_{r-1}} = \left(1-\frac{\Delta a_r}{a_r}\right)\frac{\lambda_r}{\mu_r}, \quad \text{by (1·30)}$$

$$= \left\{1+o\left(\frac{1}{r}\right)\right\} c\left\{1+\frac{b}{r}+O\left(\frac{1}{r^2}\right)\right\}, \quad \text{by (3·7)}.$$

Thus
$$\frac{u_{r-1}}{u_r} = c+O\left(\frac{1}{r}\right),$$

and the application of lemma 2 to (3·21) yields the result that

$$\Delta b_n = \frac{1}{\mu_n}\frac{a_{n-1}}{1-c}\left\{1+O\left(\frac{1}{n}\right)\right\},$$

or, in virtue of (3·34),
$$\Delta b_n = \frac{a_\infty}{1-c}\frac{1}{\mu_n}\left\{1+O\left(\frac{1}{n}\right)\right\}. \tag{3·35}$$

Since we assume that $a > \frac{1}{2}$, it follows from (3·11) that

$$\Sigma\frac{1}{n}\frac{1}{\mu_n} < \infty.$$

Thus we can write
$$b_n = \frac{a_\infty}{1-c}M_n + o(1),$$

where
$$M_n = \sum_{r=1}^{n}\mu_r^{-1} = \begin{cases} O(n^{1-a}) & \text{if } \frac{1}{2}<a<1 \\ O(\ln n) & \text{if } a=1 \\ O(1) & \text{if } a>1. \end{cases}$$

We observe that in each case
$$\frac{\Delta b_r}{b_r} = O\left(\frac{1}{\mu_r M_r}\right) = O\left(\frac{1}{r}\right).$$

Turning now to (3·22) we see that we may again apply lemma 2 because

$$\frac{b_{r-1}}{w_{r-1}}\frac{w_r}{b_r} = \left(1-\frac{\Delta b_r}{b_r}\right)\frac{\lambda_r}{\mu_r} = c+O\left(\frac{1}{r}\right).$$

Thus
$$\Delta c_n = \frac{1}{1-c}\frac{b_{n-1}}{\mu_n}\left\{1+O\left(\frac{1}{n}\right)\right\}. \tag{3·36}$$

We shall now show that
$$t'_n = b_n^2 - 2a_\infty c_n$$

is a bounded function of n. In fact, we have the identity

$$\Delta t'_n = (\Delta b_n)^2 + 2b_{n-1}\Delta b_n - 2a_\infty \Delta c_n,$$

and hence
$$\Delta t'_n = (\Delta b_n)^2 + \frac{2a_\infty}{1-c}\frac{b_{n-1}}{\mu_n}\left\{1+O\left(\frac{1}{n}\right)\right\} - \frac{2a_\infty b_{n-1}}{(1-c)\mu_n}\left\{1+O\left(\frac{1}{n}\right)\right\},$$

by (3·35) and (3·36). Thus
$$\Delta t'_n = (\Delta b_n)^2 + O\left(\frac{b_{n-1}}{n\mu_n}\right).$$

Since both
$$(\Delta b_n)^2 = O\left(\frac{1}{\mu_n^2}\right) = O\left(\frac{1}{n^{2a}}\right)$$
and
$$\frac{b_{n-1}}{n\mu_n} = O\left(\frac{M_n}{n\mu_n}\right)$$
are functions of n whose sum to infinity converges, we infer that t'_n is bounded.

On the other hand, by (3·23) and (3·34)
$$a_n^2 t_n - t'_n = 2(a_\infty - a_n)\, c_n = O(d^n c_n),$$
which tends to zero as $n \to \infty$, since it is clear from (3·36) that c_n does not increase faster than a power of n. Hence $a_n^2 t_n$ is bounded, and so is t_n because $a_n \geqslant 1$. This concludes the proof in the first case.

Next assume that the conditions (ii) are satisfied. In this case we shall be able to prove the stronger result that
$$\sum_{r=1}^{\infty} |\beta_r|^{-1} < \infty. \tag{3·37}$$
The corresponding finite sum is
$$\frac{b_n}{a_n} = \sum_{r=1}^{n} |\beta_r^{(n)}|^{-1},$$
and it is sufficient to show that (3·33) implies the boundedness of the ratio b_n/a_n.

By (3·11) and (3·12) we now have
$$w_{r-1}/\mu_r = O(r^{b-a}),$$
so that, as before,
$$\lim_{n\to\infty} a_n = a_\infty < \infty.$$
Also,
$$a_r/w_r = O(r^{-b}).$$
Hence, by lemma 3,
$$\sum_{r=0}^{n-1} \frac{a_r}{w_r} = O(n^{1-b} \ln n + 1).$$
On substituting in (3·21) we find that
$$\Delta b_n \leqslant \text{const.}\, n^{b-a}(n^{1-b}\ln n + 1)$$
$$= \text{const.}\left(\frac{\ln n}{n^{a-1}} + \frac{1}{n^{a-b}}\right),$$
which implies that b_n and hence also b_n/a_n is bounded. This establishes (3·37) by an argument analogous to that used on p. 350. It is now clear that at most a finite number of β's are, in absolute value, less than unity. Therefore (3·37) entails that
$$\sum_{r=1}^{\infty} |\beta_r|^{-2} < \infty.$$
This concludes the proof of the theorem.

Corollary 1. *The quantities*
$$\alpha_1, \alpha_2, \ldots; \quad \beta_1, \beta_2, \ldots$$
defined in (3·1) *and* (3·2) *are all non-zero* (*in fact, negative*).

Corollary 2. *If* $\gamma \geqslant 2$, *then*
$$\sum_{r=1}^{\infty} |\alpha_r|^{-\gamma} < \infty, \quad \sum_{r=1}^{\infty} |\beta_r|^{-\gamma} < \infty.$$

The conditions of theorem 5 are only sufficient for the convergence of the series (3·13). There are other processes, even analytical ones, which also have this property. We mention the following result:

THEOREM 6. *An analytical process has the property that*

$$\sum_{r=1}^{\infty} |\alpha_r|^{-2} < \infty$$

if $\lambda_0 = 0$ *and if either* (i) $c \neq 1 \quad (c > 0), \quad a > \frac{1}{2}$

or (ii) $c = 1, \quad a > 2.$

Note that no restrictions are imposed on the parameter b. Thus, when $\lambda_0 = 0$, the sum of the $|\alpha|^{-2}$ converges for a much wider range of the parameters than does the sum of the $|\beta|^{-2}$.

We omit the proof of theorem 6, as its conclusions will not be required in the rest of this paper.

10. *The limits of* $\theta_r^{(n)}$ *and* $\eta_r^{(n)}$

In constructing solutions of our problem by letting n tend to infinity in (1·48) and (1·79) no difficulty is encountered with such terms as

$$\exp(t\alpha_r^{(n)}), \quad \phi_i(\alpha_r^{(n)}), \quad \phi_i(\beta_r^{(n)}),$$

which simply become $\exp(t\alpha_r), \quad \phi_i(\alpha_r), \quad \phi_i(\beta_r),$

respectively. On the other hand, it appears that the quantities introduced in (1·43) and (1·70), namely,

$$\theta_r^{(n)} = \sum_{s=0}^{n} l_{s-1} m_s \{\phi_{s-1}(\alpha_r^{(n)})\}^2 \tag{3·38}$$

and

$$\left.\begin{aligned} \eta_0^{(n)} &= \lambda_0 + \lambda_0^2 \sum_{s=1}^{n} \frac{w_{s-1}}{\mu_s} \quad (\lambda_0 > 0) \\ \eta_r^{(n)} &= \sum_{s=0}^{n} l_{s-1} m_s \{\phi_{s-1}(\beta_r^{(n)})\}^2 \quad (r \geq 1) \end{aligned}\right\} \tag{3·39}$$

tend to a finite limit, as $n \to \infty$, only if certain restrictions are imposed on the λ's and μ's. For this purpose it is necessary to have rather delicate estimates of $\phi_s(\alpha_r^{(n)})$ and $\phi_s(\beta_r^{(n)})$. We begin with two lemmas, the first of which is essentially what is known in the literature as *Tannery's theorem* (Bromwich 1908). But for the sake of subsequent applications we shall state it here in a slightly more general form.

LEMMA 4. *Let* $W(n,t) = \sum_{s=0}^{n} v_s(n,t) \quad (n = 0, 1, 2, \ldots; t \geq 0),$

and suppose that (i) $v_s(n,t) \to V_s(t)$ *uniformly for* $t \geq 0$

and (ii) $|v_s(n,t)| \leq M_s$ *for all* n $(\geq s)$ *and* t (≥ 0),

where $\Sigma M_s < \infty.$

Then $\lim_{n\to\infty} W(n,t) = \sum_{s=0}^{\infty} V_s(t),$

uniformly for $t \geq 0$.

Proof. Defining
$$v_s(n,t) = 0$$
if $s>n$, we can write
$$W(n,t) = \sum_{s=0}^{\infty} v_s(n,t). \tag{3·40}$$

The hypotheses (i) and (ii) then imply that the series (3·40) converges uniformly with respect to the variables n and t, where n ranges over all non-negative integers and t over all non-negative real numbers. It is therefore permissible to let n tend to infinity under the summation sign, which proves the lemma.

LEMMA 5. *The real numbers $x_1, x_2, \ldots, x_{n-1}$ $(n \geqslant 2)$ satisfy the equations*†
$$1/x_{s-1} = (1+a_s) - b_s x_s \quad (s = 2, 3, \ldots, n-1), \tag{3·41}$$
where b_s is positive. Suppose there exists a number $\delta > 0$ and a positive integer $S = S(\delta) < n$ such that
$$\text{(i)}\quad 0 < x_{n-1} \leqslant (1-n^{-\delta})^{-1}, \tag{3·42}$$
$$\text{(ii)}\quad a_s + s^{-\delta} \geqslant b_s\{1-(s+1)^{-\delta}\}^{-1} \quad \text{for} \quad s \geqslant S. \tag{3·43}$$
Then
$$0 < x_{s-1} \leqslant (1-s^{-\delta})^{-1} \quad \text{for} \quad s \geqslant S.$$

Proof. We use an inductive argument for the finite sequence
$$x_{n-1}, x_{n-2}, \ldots, x_s, \ldots, x_{S-1},$$
taking (3·42) as a basic of induction.

Assuming that $s \geqslant S$ and that
$$x_s \leqslant \{1-(s+1)^{-\delta}\}^{-1},$$
we deduce from (3·41) that
$$1/x_{s-1} \geqslant (1+a_s) - b_s\{1-(s+1)^{-\delta}\}^{-1},$$
and hence by (3·43)
$$1/x_{s-1} \geqslant 1 + a_s - (a_s + s^{-\delta}) = 1 - s^{-\delta},$$
which completes the induction.

THEOREM 7. *In an analytical process whose parameters satisfy either*
$$\text{(i)}\quad 0 < c < 1, \quad a > 0 \tag{3·44}$$
or
$$\text{(ii)}\quad c = 1, \quad a > 2, \quad a - b > 1, \tag{3·45}$$
the quantities $\theta_r^{(n)}$, $\eta_r^{(n)}$ tend to finite limits as $n \to \infty$; in fact
$$\left.\begin{aligned} \lim_{n\to\infty} \theta_0^{(n)} &= \theta_0 = 1, \quad \text{if} \quad \lambda_0 = 0, \\ \lim_{n\to\infty} \theta_r^{(n)} &= \theta_r = \sum_{s=0}^{\infty} l_{s-1} m_s \{\phi_{s-1}(\alpha_r)\}^2, \end{aligned}\right\} \tag{3·46}$$
for $r \geqslant 0$ if $\lambda_0 > 0$, and for $r \geqslant 1$ if $\lambda_0 = 0$, and
$$\left.\begin{aligned} \lim_{n\to\infty} \eta_0^{(n)} &= \eta_0 = 1, \quad \text{if} \quad \lambda_0 = 0, \\ \lim_{n\to\infty} \eta_r^{(n)} &= \sum_{s=0}^{\infty} l_{s-1} m_s \{\phi_{s-1}(\beta_r^{(n)})\}^2, \end{aligned}\right\} \tag{3·47}$$
or $r \geqslant 0$ if $\lambda_0 > 0$ and for $r \geqslant 1$ if $\lambda_0 = 0$.

† The a_s and b_s bear no relation to the coefficients occurring in (3·14).

Proof. As far as possible we shall discuss (3·46) and (3·47) together. The recurrence formula (1·18) can be written

$$\frac{1}{z_{s-1}(\xi)} = \left(1+\frac{\lambda_s}{\mu_s}+\frac{\xi}{\mu_s}\right)-\frac{\lambda_s}{\mu_s}z_s(\xi), \tag{3·48}$$

where
$$z_s(\xi) = \frac{\phi_s(\xi)}{\lambda_s\phi_{s-1}(\xi)} \quad (s = 1, 2, \ldots). \tag{3·49}$$

It is easily seen from (1·18) that consecutive polynomials $\phi_{s-1}(\xi)$, $\phi_s(\xi)$ cannot vanish simultaneously for a non-zero value of ξ.

We are interested in the values of $z_s(\xi)$ when $\xi = \alpha_r^{(n)}$ or $\xi = \beta_r^{(n)}$. The suffix r remains unaltered throughout this proof, and it is important to note that, by (3·3) and (3·4),

$$|\alpha_r^{(n)}| < |\alpha_r|, \quad |\beta_r^{(n)}| < |\beta_r| \quad (n = r, r+1, \ldots),$$

so that we may assume that
$$|\xi| \leqslant K.$$

where K is a constant (independent of n). In order to link (3·48) with lemma 5 we put

$$a_s = \frac{\lambda_s}{\mu_s}+\frac{\xi}{\mu_s}, \quad b_s = \frac{\lambda_s}{\mu_s},$$

and we shall verify that (3·42) and (3·43) are satisfied for sufficiently great n provided δ is suitably chosen. It is necessary to discuss the two alternative sets of conditions (3·44) and (3·45) separately.

(i) Assuming first that (3·44) is true we choose a number δ such that

$$0 < \delta < \min(a, 1). \tag{3·50}$$

Testing (3·43) we find, after some calculations, that

$$a_s+s^{-\delta}-b_s\{1-(s+1)^{-\delta}\} > a_s+s^{-\delta}-b_s(1-s^{-\delta})^{-1} = s^{-\delta}\left[1-\frac{\lambda_s}{\mu_s}(1+s^{-\delta}+\ldots)+\frac{\xi s^{\delta}}{\mu_s}\right].$$

The expression in the square brackets tends to $1-c>0$, as $s\to\infty$, since $\lambda_s/\mu_s \to c$ by (3·8) and $\xi s^{\delta}/\mu_s \to 0$ by (3·11) and (3·50). Moreover, since the bound for ξ is independent of n, there exists an integer $S = S(\delta)$, independent of n, such that

$$a_s+s^{-\delta} \geqslant b_s\{1-(s+1)^{-\delta}\}^{-1} \quad \text{for} \quad s \geqslant S.$$

This is the condition (3·43) of lemma 5.

We now turn to the verification of (3·42). Since, by definition of $\alpha_r^{(n)}$, $\phi(\alpha_r^{(n)}) = 0$ and hence $z_n(\alpha_r^{(n)}) = 0$, we have, by (3·48), that

$$z_{n-1}(\alpha_r^{(n)}) = \left(1+\frac{\lambda_n}{\mu_n}+\frac{\alpha_r^{(n)}}{\mu_n}\right)^{-1}.$$

As $n\to\infty$, the expression on the right tends to $(1+c)^{-1}<1$. Hence we can find an integer $n_1 = n_1(\delta)$ such that
$$0 < z_{n-1}(\alpha_r^{(n)}) < (1-n^{-\delta})^{-1} \quad \text{for} \quad n \geqslant n_1.$$

This agrees with (3·42), and lemma 5 may be applied with

$$x_{n-1} = z_{n-1}(\alpha_r^{(n)}).$$

On the other hand, when $\xi = \beta_r^{(n)}$, and therefore $\psi_n(\beta_r^{(n)}) = 0$, we deduce from (1·54) that

$$z_n(\beta_r^{(n)}) = 1,$$

and hence, by (3·48), $$z_{n-1}(\beta_r^{(n)}) = \left(1+\frac{\beta_r^{(n)}}{\mu_n}\right)^{-1}.$$

Since $|\beta_r^{(n)}| \leqslant K$ for all n, and $\mu_n \sim \text{const.}\, n^a$, where $a > \delta$ there exists an integer $n_2 = n_2(\delta)$ such that
$$0 < z_{n-1}(\beta_r^{(n)}) < (1-n^{-\delta})^{-1}$$
for $n \geqslant n_2$. Hence we may apply lemma 5 with $x_s = z_s(\beta_r^{(n)})$ $(s = 1, 2, \ldots, n-1)$.

The results so far obtained may be summarized as follows:

Let
$$N = \max(r, S, n_1, n_2) = N(\delta).$$
Then
$$0 < \frac{\phi_{s-1}(\xi)}{\lambda_{s-1}\phi_{s-2}(\xi)} < (1-s^{-\delta})^{-1} \tag{3·51}$$
if $n \geqslant s \geqslant N$, where $\xi = \alpha_r^{(n)}$ or $\beta_r^{(n)}$. If in (3·51) we replace s by u and multiply the inequalities for $u = N, N+1, \ldots, s$, we find that
$$\phi_{s-1}(\xi) = H l_{s-1}^{-1} \prod_{u=2}^{s} (1-u^{-\delta})^{-1}, \tag{3·52}$$
where the constant H depends only on N. Since $\delta < 1$, we have
$$\prod_{u=2}^{s}(1-u^{-\delta})^{-1} = \exp\left\{-\sum_{u=2}^{s} \ln(1-u^{-\delta})\right\}$$
$$\leqslant \exp\left(k' \sum_{u=2}^{s} u^{-\delta}\right) \leqslant \exp(ks^{1-\delta}),$$
where k' and k are certain constants.

Hence, finally, it follows that the general term in (3·38) or (3·39) is
$$\begin{aligned} M_s &= l_{s-1} m_s \{\phi_{s-1}(\xi)\}^2 \\ &< \text{const.}\frac{w_s}{\lambda_s}\exp(2ks^{1-\delta}) \\ &< \text{const.}\, s^{-a} c^s s^b \exp(2ks^{1-\delta}) \\ &< \text{const.}\exp\{s\ln c + (b-a)\ln s + 2ks^{1-\delta}\}. \end{aligned}$$
Since in the present case $\ln c < 0$, we have
$$\Sigma M_s < \infty,$$
and the application of lemma 4 immediately leads to (3·46) and (3·47).

(ii) Under the second set of conditions (3·45) the foregoing proof has to be modified in some points. We now choose δ such that
$$\max(1, b) < \delta < a-1. \tag{3·53}$$
The verification of (3·43) runs as follows:
$$\begin{aligned} a_s + s^{-\delta} - b_s\{1-(s+1)^{-\delta}\}^{-1} &= (a_s - b_s) + \frac{1}{s^\delta} - b_s\left\{\frac{1}{(s+1)^\delta} + \frac{1}{(s+1)^{2\delta}} + \ldots\right\} \\ &= \frac{\xi}{\mu_s} + \frac{1}{s^\delta} - \left(1 + \frac{b}{s} + O\left(\frac{1}{s^2}\right)\right)\left\{\frac{1}{(s+1)^\delta} + \ldots\right\}, \end{aligned}$$
and, since $\xi/\mu_s = O(s^{-a})$, this may be written as
$$\frac{1}{s^\delta} - \frac{1}{(s+1)^\delta} - \frac{b}{s(s+1)^\delta} + O\left(\frac{1}{s^a}\right) + O\left(\frac{1}{s^{2\delta}}\right) + O\left(\frac{1}{s^{2+\delta}}\right) = \frac{\delta - b}{s(s+1)^\delta} + O\left(\frac{1}{s^a}\right) + O\left(\frac{1}{s^{2\delta}}\right) + O\left(\frac{1}{s^{2+\delta}}\right).$$

In order that the first term should be positive and dominate the others, we must have

$$\delta-b>0, \quad 1+\delta<a, \quad 1+\delta<2\delta.$$

These three conditions are in fact consequences of (3·53).

The argument then proceeds as in (i) and leads to (3·52). Since we now have $\delta>1$, the product remains bounded as $s\to\infty$, so that we can write

$$\phi_{s-1}(\xi)<H' l_{s-1}^{-1},$$

where H' is a constant. It follows that the general term in (3·38) or in (3·39) is majorized by

$$M'_s = \text{const.}\, s^{b-a},$$

whence

$$\Sigma M'_s<\infty,$$

because $a-b>1$. This concludes the proof of the theorem.

In theorem 7 the complete and modified sections have been treated together. If we confine ourselves to the complete sections, the result can be established under wider conditions. We mention without proof the following:

THEOREM 8. *An analytical process in which*

$$c>1, \quad a>0$$

has the property that

$$\lim_{n\to\infty} \theta_r^{(n)} = \theta_r = \sum_{s=0}^{\infty} l_{s-1} m_s \{\phi_{s-1}(\alpha_r)\}^2,$$

except when $\lambda_0 = r = 0$, *in which case* $\theta_0 = 1$.

11. *The construction of solutions*

We shall now show that under certain conditions

$$\lim_{n\to\infty} f_{ij}^{(n)}(t) \quad \text{and} \quad \lim_{n\to\infty} g_{ij}^{(n)}(t)$$

exist and furnish solutions of our problem. Throughout the next two sections we confine ourselves to analytical processes to which theorems 5 and 7 apply, i.e. we assume that either (3·32) or (3·33) is fulfilled. Each of these conditions implies that

$$\sum_s (w_{s-1}/\mu_s)<\infty. \tag{3·54}$$

Our argument is based on lemma 4, which, in the first instance, is applied to (1·48), viz.

$$f_{ij}^{(n)}(t) = \sum_{r=0}^{n} \exp(t\alpha_r^{(n)})\, d_r^{(n)}(i,j).$$

It is clear that the general term tends to a finite limit as $n\to\infty$, thus

$$\lim_{n\to\infty} \exp(t\alpha_r^{(n)})\, d_r^{(n)}(i,j) = \exp(t\alpha_r)\, d_r(i,j),$$

say. By (1·45), (1·41) and (1·46) it is found that

$$d_0(i,j) = \zeta_i \delta_{0j} \quad \text{if} \quad \lambda_0 = 0, \tag{3·55}$$

where

$$\zeta_0 = 1, \quad \zeta_i = 1-\left(\sum_{s=0}^{i-1} w_s^{-1}\right)\left(\sum_{s=0}^{\infty} w_s^{-1}\right)^{-1}, \tag{3·56}$$

and $$d_r(i,j) = l_{i-1} m_j \phi_{i-1}(\alpha_r)\, \phi_{j-1}(\alpha_r)/\theta_r$$
in all other cases.

Next we have to establish estimates of the form

$$|\exp(t\alpha_r^{(n)})\, d_r^{(n)}(i,j)| \leqslant M_r(i,j)$$

(for sufficiently great r), where

$$\sum_r M_r(i,j) < \infty.$$

Assume first that i is arbitrary but that $j \geqslant 1$, and write (1·46) in the form

$$d_r^{(n)}(i,j) = l_{i-1} m_j \frac{\phi_{i-1}(\alpha_r^{(n)})}{(\theta_r^{(n)})^{\frac{1}{2}}} \frac{\phi_{j-1}(\alpha_r^{(n)})}{(\theta_r^{(n)})^{\frac{1}{2}}}.$$

Retaining only one term in the expansion (1·43) we obtain the inequalities

$$\theta_r^{(n)} > l_{s-1} m_s \{\phi_{s-1}(\alpha_r^{(n)})\}^2, \tag{3·57}$$

where $s = 0, 1, 2, \ldots, n$. On taking in turn $s = i+3$ and $s = j$, we find that

$$|d_r^{(n)}(i,j)| \leqslant \tfrac{1}{2} k_i (w_{j-1}/\mu_j)^{\frac{1}{2}} \left| \frac{\phi_{i-1}(\alpha_r^{(n)})}{\phi_{i+2}(\alpha_r^{(n)})} \right|,$$

where $\frac{1}{2}k_i = l_{i-1}(l_{i+2} m_{i+3})^{-\frac{1}{2}}$ is a constant depending on i only. In order to examine the last factor further let

$$g(\xi) = \left| \frac{\xi^3 \phi_{i-1}(\xi)}{\phi_{i+2}(\xi)} \right|.$$

Since $g(\xi) \to 1$ as $|\xi| \to \infty$, there exists a number Ξ such that

$$g(\xi) < 2 \quad \text{if} \quad |\xi| \geqslant \Xi.$$

But $|\alpha_r| \to \infty$, and hence we can find a positive integer R_i such that $|\alpha_r| \geqslant \Xi$, if $r \geqslant R_i$. By (3·3) we then have *a fortiori* that $|\alpha_r^{(n)}| \geqslant \Xi$, if $r \geqslant R_i$ (independently of n), and therefore

$$\left| \frac{\phi_{i-1}(\alpha_r^{(n)})}{\phi_{i+2}(\alpha_r^{(n)})} \right| < 2\,|\alpha_r^{(n)}|^{-3} \leqslant 2\,|\alpha_r|^{-3}.$$

Hence, finally, since $\exp(t\alpha_r^{(n)}) \leqslant 1$ for $t \geqslant 0$,

$$|\exp(t\alpha_r^{(n)})\, d_r^{(n)}(i,j)| < k_i (w_{j-1}/\mu_j)^{\frac{1}{2}}\, |\alpha_r|^{-3} = M_r(i,j) \quad (j \geqslant 1),$$

say.

When $j = 0$, (1·46) becomes

$$d_r^{(n)}(i,0) = l_{i-1} \phi_{i-1}(\alpha_r^{(n)})/\theta_r^{(n)},$$

whence by (3·57) with $s = i+2$,

$$|\exp(t\alpha_r^{(n)})\, d_r^{(n)}(i,0)| < \text{const.} \left| \frac{\phi_{i-1}(\alpha_r^{(n)})}{\{\phi_{i+1}(\alpha_r^{(n)})\}^2} \right| < \text{const.}\, |\alpha_r|^{-4} = M_r(i,0),$$

where the constant depends only on i. In each case we have

$$\sum_r M_r(i,j) < \infty.$$

We observe that, on summing over r, we get that

$$f_{ij}^{(n)}(t) < K_i (w_{j-1}/\mu_j)^{\frac{1}{2}} \quad (i \geqslant 0, j \geqslant 1). \tag{3·58}$$

Interchanging the roles of i and j we derive the analogous result

$$f_{ij}^{(n)}(t) < K'_j(w_{i-1}/\mu_i)^{\frac{1}{2}} \quad (i \geq 1, j \geq 0), \tag{3·59}$$

where K_i, K'_j are constants depending only on i and j respectively. The application of lemma 4 now entails that

$$\lim_{n\to\infty} f_{ij}^{(n)}(t) = \sum_{r=0}^{\infty} \exp(t\alpha_r)\, d_r(i,j) = f_{ij}(t), \tag{3·60}$$

the series converging uniformly for $t \geq 0$. The series obtained by formal differentiation of (3·60) also converges uniformly, since its general term is majorized by

$$|\, k_i(w_{j-1}/\mu_j)^{\frac{1}{2}}\, \alpha_r^{-2} \,|$$

(with a slight modification when $j = 0$) whose sum with respect to r converges. Hence

$$\lim_{n\to\infty} \frac{d}{dt} f_{ij}^{(n)}(t) = \sum_{r=0}^{\infty} \alpha_r \exp(\alpha_r t)\, d_r(i,j) = \frac{d}{dt} f_{ij}(t).$$

The finite matrix $F^{(n)}(t)$ satisfies (1·11), which implies that

$$\frac{d}{dt} f_{ij}^{(n)}(t) = \lambda_{j-1} f_{i,j-1}^{(n)}(t) - (\lambda_j + \mu_j) f_{ij}^{(n)}(t) + \mu_{j+1} f_{i,j+1}^{(n)}(t)$$

$(i, j = 0, 1, \ldots, n-1)$. Letting n tend to ∞ for fixed i and j we see that the infinite matrix

$$F(t) = \sum_{r=0}^{\infty} \exp(t\alpha_r)\, D_r,$$

where

$$D_r = (d_r(i,j))$$

is a solution of our problem, i.e.

$$\frac{d}{dt} F(t) = F(t)\, A.$$

Starting from (1·85) we can prove by an analogous argument that $F(t)$ satisfies the backward equation

$$\frac{d}{dt} F(t) = AF(t).$$

From (1·14), (1·15) and (1·16) we deduce that

$$f_{ij}(t) \geq 0 \quad (t \geq 0),$$

$$f_{ij}(0) = \delta_{ij}$$

and

$$\sum_{j=0}^{\infty} f_{ij}(t) \leq 1.$$

Moreover, the matrix $F(t)$ satisfies the Chapman-Kolmogorov equation

$$F(t+\tau) = F(t)\, F(\tau) \quad (t \geq 0, \tau \geq 0). \tag{3·61}$$

Indeed, we have seen (cf. (1·83)) that this equation is satisfied by the finite matrix $F^{(n)}(t)$ thus

$$\sum_{s=0}^{n} f_{is}^{(n)}(t) f_{sj}^{(n)}(\tau) = f_{ij}^{(n)}(t+\tau). \tag{3·62}$$

By (3·58) and (3·59) we have

$$0 \leq f_{is}^{(n)}(t) f_{sj}^{(n)}(\tau) < K_i K_j (w_{s-1}/\mu_s) \quad (s \geq 0),$$

and since we are assuming (see (3·54)) that

$$\sum_s (w_{s-1}/\mu_s) < \infty$$

(3·61) follows from (3·62) by lemma 4. A similar limiting process may be applied to

$$g_{ij}^{(n)}(t) = \sum_{r=0}^{n} \exp(t\beta_r^{(n)})\, e_r^{(n)}(i,j)$$

(see (1·79) and (1·75)), the quantities $\beta_r^{(n)}$ now taking the place of the $\alpha_r^{(n)}$. This leads to a solution

$$G(t) = (g_{ij}(t)), \tag{3·63}$$

where

$$g_{ij}(t) = \sum_{r=0}^{\infty} \exp(t\beta_r)\, e_r(i,j),$$

$$e_0(i,j) = \begin{cases} \lambda_0^2 m_j (l_{j-1}\eta_0)^{-1} & \text{if} \quad \lambda_0 > 0, \\ \delta_{0j} & \text{if} \quad \lambda_0 = 0, \end{cases} \tag{3·64}$$

and

$$e_r(i,j) = l_{i-1} m_j \phi_{i-1}(\beta_r)\, \phi_{j-1}(\beta_r)/\eta_r \quad (r \geqslant 1).$$

The solution (3·63) has the properties that

$$\left.\begin{aligned} 0 \leqslant g_{ij}(t) \leqslant 1, \\ g_{ij}(0) = \delta_{ij} \\ \sum_j g_{ij}(t) \leqslant 1. \end{aligned}\right\} \tag{3·65}$$

and

It is interesting to observe that the equality sign holds in the last formula if we impose certain conditions on the parameters of the process, which are somewhat more restrictive than those stated at the beginning of this section.

THEOREM 9. *If either* (i) $c < 1, \quad a > \frac{1}{2}$ (3·66)

or (ii) $c = 1, \quad a > 2, \quad a - b > 2,$ (3·67)

then

$$\sum_{j=0}^{\infty} g_{ij}(t) = 1 \tag{3·68}$$

$(t \geqslant 0, i = 0, 1, \ldots)$.

Proof. The inequality analogous to (3·58) is

$$g_{ij}^{(n)}(t) < L_i (w_{j-1}/\mu_j)^{\frac{1}{2}},$$

say, so that (3·68) follows from (1·82) by lemma 4, provided

$$\Sigma (w_{j-1}/\mu_j)^{\frac{1}{2}} < \infty.$$

This is certainly true if (3·66) or (3·67) holds.

We have seen then that both $F(t)$ and $G(t)$ are solutions satisfying the same initial conditions and having all the properties which they can reasonably be expected to possess. One therefore wonders whether the solutions are in fact identical, as they would certainly be if the system were finite. We have shown elsewhere (D, theorems 6 to 8) that the problem has in fact a unique solution in a wide range of cases. For analytical processes, non-uniqueness can occur only when

$$c = 1, \quad a > 2, \quad 1 < b < a.$$

We shall prove in the next section that the solutions $F(t)$ and $G(t)$ do in fact differ for a certain range of the parameters.

12. *Asymptotic values*

Subject to the conditions stated in the preceding section the series

$$F(t) = \sum_{r=0}^{\infty} \exp(t\alpha_r)\, D_r \tag{3·69}$$

and

$$G(t) = \sum_{r=0}^{\infty} \exp(t\beta_r)\, E_r \tag{3·70}$$

converge uniformly for $t \geqslant 0$, and their values when $t \to \infty$ may be found by letting t tend to ∞ in each term. Since all α's and β's are negative with the exception of β_0, which is always zero, and with the possible exception of α_0 which may be zero or negative, but which is certainly zero when $\lambda_0 = 0$, we have that

$$F(\infty) = \begin{cases} 0 & \text{if} \quad \alpha_0 < 0, \\ D_0 & \text{if} \quad \alpha_0 = 0, \end{cases} \tag{3·71}$$

$$G(\infty) = E_0. \tag{3·72}$$

The explicit values are given in (3·55) and (3·64), thus

$$F(\infty) = \begin{pmatrix} 1 & 0 & 0 & . & . & . \\ \zeta_1 & 0 & 0 & . & . & . \\ \zeta_2 & 0 & 0 & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (\lambda_0 = 0) \tag{3·73}$$

and

$$G(\infty) = \frac{1}{\eta_0} \begin{pmatrix} 1 & \lambda_0 l_0/m_1 & \lambda_0 l_1/m_2 & \lambda_0 l_2/m_3 & . & . & . \\ 1 & \lambda_0 l_0/m_1 & \lambda_0 l_1/m_2 & \lambda_0 l_2/m_3 & . & . & . \\ 1 & \lambda_0 l_0/m_1 & \lambda_0 l_1/m_2 & \lambda_0 l_2/m_3 & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (\lambda_0 > 0) \tag{3·74}$$

if $\lambda_0 > 0$, and

$$G(\infty) = \begin{pmatrix} 1 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (\lambda_0 = 0) \tag{3·75}$$

if $\lambda_0 = 0$. These results may be compared with the general conclusions about asymptotic values expounded in chapter II, § 6.

When $\Sigma w_s^{-1} = \infty$, the formula (3·56) for ζ_i should be interpreted as $\zeta_i = 1$. In that case $F(\infty) = G(\infty)$.

But if $\lambda_0 = 0$ *and*

$$\Sigma w_s^{-1} < \infty \tag{3·76}$$

the two solutions have different asymptotic values and are therefore distinct. Since

$$w_s^{-1} \sim \text{const.}\, c^{-s} s^{-b}$$

(3·76) is compatible with the condition of theorems 5, 7 and 9 if

$$c = 1, \quad a > 2, \quad b > 1, \quad a - b > 2. \tag{3·77}$$

For example, we can take $a = 5, \quad b = 2, \quad c = 1;$

a simple process with these parameters is given by

$$\lambda_n = \mu(n^5 + 2n^4), \quad \mu_n = \mu n^5.$$

We summarize these results in

THEOREM 10. *If the parameters of an analytical process in which $\lambda_0 = 0$, satisfy the conditions* (3·77) *then the matrices $F(t)$ and $G(t)$ are distinct and in fact have the following properties:*

$$\frac{d}{dt}F(t) = F(t)\,A = AF(t),$$

$$\frac{d}{dt}G(t) = G(t)\,A = AG(t),$$

$$F(0) = G(0) = I,$$

$$0 \leqslant f_{ij}(t) \leqslant 1, \quad \sum_{j=0}^{\infty} f_{ij}(t) \leqslant 1,$$

$$0 \leqslant g_{ij}(t) \leqslant 1, \quad \sum_{j=0}^{\infty} g_{ij}(t) = 1,$$

$$F(t+\tau) = F(t)\,F(\tau), \quad G(t+\tau) = G(t)\,G(\tau),$$

but

$$f_{i0}(\infty) = \zeta_i < 1 \quad (i > 0), \quad g_{i0}(\infty) = 1.$$

We recall that the ith row of $F(t)$ or $G(t)$ represents a solution of the equation

$$\frac{d}{dt}p(t) = p(t)\,A$$

with the initial values

$$p_i(0) = 1, \quad p_k(0) = 0 \quad (k \neq i).$$

Since all the rows of (3·74) or of (3·75) are identical, the solution $G(t)$ (if it exists) has the 'ergodic property' (see p. 344) because the asymptotic distribution does not depend on the initial distribution. On the other hand, if $\zeta_i < 1$, $F(t)$ is not ergodic since the rows of (3·73) are then distinct; in these circumstances, if the system remains finite at all, it can only tend to the state of complete annihilation. However, the probability that this should happen depends on the distribution at time $t = 0$. The spectral resolution of the solution (3·69) or (3·70) gives an indication of the speed with which the ultimate distribution is reached. Thus we may write

$$F(t) = D_0 + \exp(t\alpha_1)\,M(t) \quad (\lambda_0 = 0)$$

$$G(t) = E_0 + \exp(t\beta_1)\,N(t),$$

where the coefficients of $M(t)$ and $N(t)$ remain finite as $t \to \infty$. Thus the greater $|\alpha_1|$ (or $|\beta_1|$), the more rapidly will $F(t)$ (or $G(t)$) approach D_0 (or E_0). Since in all cases considered by us

$$\sum_{r=1}^{\infty} |\alpha_r|^{-2} \leqslant \sum_{r=1}^{\infty} |\beta_r|^{-2} \leqslant h,$$

we obtain the estimate

$$|\alpha_1| \geqslant |\beta_1| \geqslant h^{-\frac{1}{2}}.$$

Hence

$$|f_{ij}(t) - f_{ij}(\infty)| \leqslant \text{const.}\exp(-h^{-\frac{1}{2}}t).$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Chapter IV. Examples

13. *Constant coefficients* ($\lambda_0 = 0$)

As an illustration of the general theory of chapter II, we consider the case when λ_n and μ_n are constant (for $n \geq 1$). We begin with the case $\lambda_0 = 0$, which turns out to be simpler than $\lambda_0 > 0$. So let

$$\lambda_0 = 0, \qquad \lambda_n = \lambda > 0 \quad (n \geq 1), \qquad \mu_n = \mu > 0 \quad (n \geq 1). \tag{4·1}$$

Then, in the notation of §5, chapter II, we have

$$\phi_0(x) = 1, \quad \phi_1(x) = x + \lambda + \mu,$$

$$\bar{\phi}_n(x) - (x + \lambda + \mu)\,\phi_{n-1}(x) + \lambda\mu\phi_{n-2}(x) = 0 \quad (n \geq 2).$$

If we put

$$x = -(\lambda + \mu) + 2\sqrt{(\lambda\mu)}\cos\omega, \tag{4·2}$$

$$\bar{\phi}_n(x) = [\sqrt{(\lambda\mu)}]^n\,\Phi_n(\omega), \tag{4·3}$$

we find that

$$\Phi_n(\omega) = \frac{\sin(n+1)\,\omega}{\sin\omega}. \tag{4·4}$$

Hence $\Phi_n(\omega)$ vanishes at $\omega = \omega_r^{(n)}$ $(r = 1, \ldots, n)$, where

$$\omega_r^{(n)} = \frac{r\pi}{n+1}, \tag{4·5}$$

and hence

$$\alpha_r^{(n)} = -(\lambda + \mu) + 2\sqrt{(\lambda\mu)}\cos\omega_r^{(n)}. \tag{4·6}$$

By inserting these results in the expressions (1·43) and (2·25) for $\bar{\theta}_r^{(n)}$, we obtain

$$\bar{\theta}_r^{(n)} = -\frac{\alpha_r^{(n)}}{\mu}\sum_{\nu=1}^{n}\left(\frac{\sin\nu\omega_r^{(n)}}{\sin\omega_r^{(n)}}\right)^2,$$

and on evaluating the trigonometric sum involved,

$$\frac{1}{\bar{\theta}_r^{(n)}} = -\frac{2\mu}{\pi}\frac{\sin^2\omega_r^{(n)}}{\alpha_r^{(n)}}\frac{\pi}{n+1}. \tag{4·7}$$

If we write $\Delta\omega = \dfrac{\pi}{n+1}\left(= \omega_{r+1}^{(n)} - \omega_r^{(n)}\right)$, we see that

$$\frac{1}{\bar{\theta}_r^{(n)}} = -\frac{2\mu}{\pi}\frac{\sin^2\omega_r^{(n)}}{\alpha_r^{(n)}}\Delta\omega,$$

and by a formal limiting process (which is easily justified)

$$\mathrm{d}\bar{\rho}(x) = -\frac{2\mu}{\pi}\frac{\sin^2\omega}{x}\,\mathrm{d}\omega, \tag{4·8}$$

so that the spectrum is continuous and extends over the interval

$$-(\lambda + \mu) - 2\sqrt{(\lambda\mu)} \leq x \leq -(\lambda + \mu) + 2\sqrt{(\lambda\mu)}. \tag{4·9}$$

It is interesting to observe (cf. the remarks in §7, chapter II) how the continuous spectrum arises: the points $\alpha_r^{(n)}$ fill out the interval (4·9) more and more densely as $n \to \infty$, whilst the discontinuities $1/\bar{\theta}_r^{(n)}$ tend to zero; moreover, for fixed r, $\alpha_r^{(n)}$ tends to the right-hand end-point

$x = -(\lambda+\mu)+2\surd(\lambda\mu)$ of the spectrum. Using (4·8), the explicit formulae for $f_{ij}(t)$ are found to be

$$\left.\begin{aligned} f_{ij}(t) &= \frac{2}{\pi}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-j} \mathrm{e}^{-(\lambda+\mu)t}\int_0^\pi \sin i\omega \sin j\omega\, \mathrm{e}^{2t\surd(\lambda\mu)\cos\omega}\,\mathrm{d}\omega \quad (i\geqslant 1, j\geqslant 1),\\ f_{0j}(t) &= \delta_{0j} \quad (j\geqslant 0),\\ f_{i0}(t) &= \zeta_i - \frac{2}{\pi}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-1} \mathrm{e}^{-(\lambda+\mu)t}\int_0^\pi \sin i\omega\, \mathrm{e}^{2t\surd(\lambda\mu)\cos\omega}\frac{\mu \sin\omega\,\mathrm{d}\omega}{\lambda+\mu-2\surd(\lambda\mu)\cos\omega} \quad (i\geqslant 1), \end{aligned}\right\} \tag{4·10}$$

with

$$\zeta_i = \begin{cases} 1 & \text{if } \lambda\leqslant\mu, \\ \left(\dfrac{\mu}{\lambda}\right)^i & \text{if } \lambda>\mu. \end{cases} \tag{4·11}$$

The integrals occurring in (4·10) can be expressed in terms of the Bessel functions I_m (Magnus & Oberhettinger 1943, pp. 19, 26), given by

$$I_m(z) = \frac{1}{\pi}\int_0^\pi \cos m\omega\, \mathrm{e}^{z\cos\omega}\,\mathrm{d}\omega. \tag{4·12}$$

For $i\geqslant 1$ and $j\geqslant 1$, the result is

$$f_{ij}(t) = \left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-j}\mathrm{e}^{-(\lambda+\mu)t}\{I_{i-j}[2\surd(\lambda\mu)\,t]-I_{i+j}[2\surd(\lambda\mu)\,t]\}. \tag{4·13}$$

For $j = 1$, this simplifies to

$$f_{i1}(t) = \frac{i}{\mu}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i}\mathrm{e}^{-(\lambda+\mu)t}\frac{I_i[2\surd(\lambda\mu)\,t]}{t},$$

and since

$$f'_{i0}(t) = \mu f_{i1}(t), \quad f_{i0}(0) = 0 \quad (i\geqslant 1),$$

$$f_{i0}(t) = i\left(\sqrt{\frac{\mu}{\lambda}}\right)^{i}\int_0^t \mathrm{e}^{-(\lambda+\mu)\tau} I_i[2\surd(\lambda\mu)\,\tau]\frac{\mathrm{d}\tau}{\tau}. \tag{4·14}$$

The work of this section can be used to calculate the distribution of 'busy periods' in a queue with Poissonian input and exponential service-time. This problem has also been considered by Kendall (1951), who sketches a derivation, by an entirely different method, of a formula for $f'_{10}(t)$ $(=\mu f_{11}(t))$ which agrees with (4·13). Our analysis shows the result to be valid for all positive λ, μ and not only for $\lambda\leqslant\mu$, as Kendall appears to suggest. It also confirms his conjecture that, when $\lambda>\mu$, an infinitely long busy period is an event of positive probability; in fact this probability is

$$1-f_{10}(\infty) = 1-\zeta_1 = 1-\frac{\lambda}{\mu}$$

(cf. (4·10) and (4·11)).

14. *Constant coefficients* $(\lambda_0>0)$

Suppose now that

$$\lambda_n = \lambda>0 \quad (n\geqslant 0), \qquad \mu_n = \mu>0 \quad (n\geqslant 1) \tag{4·15}$$

(so that $\lambda_0>0$). We now have

$$\phi_{-1}(x) = 1, \quad \phi_0(x) = x+\lambda, \quad \phi_n(x)-(x+\lambda+\mu)\,\phi_{n-1}(x)+\lambda\mu\phi_{n-2}(x) = 0 \quad (n\geqslant 2),$$

and with $x = -(\lambda+\mu)+2\surd(\lambda\mu)\cos\omega$ as before,

$$\phi_n(x) = [\surd(\lambda\mu)]^n F_n(\omega) = \frac{[\surd(\lambda\mu)]^n}{\sin\omega}[\surd(\lambda\mu)\sin(n+2)\,\omega-\mu\sin(n+1)\,\omega]. \tag{4·16}$$

The zeros of $\phi_n(x)$ are inconvenient in calculations, so that it is preferable to work with $\psi_n(x) = x\bar{\psi}_n(x)$. It is found that

$$\bar{\psi}_n(x) = [\sqrt{(\lambda\mu)}]^n \frac{\sin(n+1)\,\omega}{\sin\omega}, \tag{4·17}$$

so that, with $\omega_r^{(n)} = \dfrac{r\pi}{n+1}$ as before,

$$\beta_0^{(n)} = 0, \quad \beta_r^{(n)} = -(\lambda+\mu) + 2\sqrt{(\lambda\mu)}\cos\omega_r^{(n)} \quad (r = 1, \ldots, n).$$

Some lengthy computations of trigonometric sums lead to

$$\eta_0^{(n)} = \lambda \frac{1-\left(\dfrac{\lambda}{\mu}\right)^{n+1}}{1-\dfrac{\lambda}{\mu}} \, [= (n+1)\,\lambda \quad \text{if} \quad \lambda = \mu],$$

$$\eta_r^{(n)} = \frac{1}{\mu}\sum_{\nu=0}^{n} [F_{\nu-1}(\omega_r^{(n)})]^2 = -\frac{\beta_r^{(n)}(n+1)}{2\sin^2\omega_r^{(n)}}.$$

Hence
$$\frac{1}{\eta_r^{(n)}} = \frac{2}{\pi}\frac{\sin^2\omega_r^{(n)}}{(-\beta_r^{(n)})}\,\Delta\omega \quad (r \geqslant 1),$$

and (formally)
$$\mathrm{d}\sigma(x) = \frac{2}{\pi}\frac{\sin^2\omega}{(-x)}\,\mathrm{d}\omega \quad (0 \leqslant \omega \leqslant \pi), \tag{4·18}$$

while, in addition, $\sigma(x)$ has a discontinuity $\sigma_0 = \lim_{n\to\infty} 1/\eta_0^{(n)}$ at $x = 0$, with

$$\sigma_0 = \begin{cases} 0 & \text{if} \quad \lambda \geqslant \mu, \\ \dfrac{\mu-\lambda}{\mu\lambda} & \text{if} \quad \lambda < \mu; \end{cases} \tag{4·19}$$

cf. (2·58). Using these results, and a certain amount of manipulation, expressions for $g_{ij}(t)$ can be derived. We quote only the final result, which is most conveniently expressed as

$$g'_{ij}(t) = \left(\sqrt{\frac{\mu}{\lambda}}\right)^{i-j} \mathrm{e}^{-(\lambda+\mu)t} \begin{bmatrix} -(\lambda+\mu)\,I_{i-j} + \sqrt{(\lambda\mu)}\,I_{i-j-1} + \sqrt{(\lambda\mu)}\,I_{i+j+1} \\ +\lambda I_{i+j+2} - 2\sqrt{(\lambda\mu)}\,I_{i+j+1} + \mu I_{i+j} \end{bmatrix}, \tag{4·20}$$

the suppressed argument in the Bessel functions I_m being $2\sqrt{(\lambda\mu)}\,t$. Expressions for $g_{ij}(t)$ are then obtained by integration, using $g_{ij}(0) = \delta_{ij}$.

The expression for $g_{ij}(t)$ simplifies remarkably when $\lambda = \mu$; it then reduces to

$$g_{ij}(t) = \mathrm{e}^{-2\lambda t}[I_{i-j}(2\lambda t) + I_{i+j+1}(2\lambda t)]. \tag{4·21}$$

We observe finally that we have worked with $f_{ij}(t)$ in §13 and with $g_{ij}(t)$ in §14 for ease of calculation; in either case, it is known that $f_{ij}(t) = g_{ij}(t)$, because the coefficients λ_j and μ_j are bounded and this ensures that the solution of (0·14) is unique (Arley 1943). We should mention that Kac (1947) has performed calculations somewhat similar to those of §§ 13, 14 (relating to Markov chains, i.e. Markov processes with discrete time). Nevertheless, our results are new as far as we are aware, and we cannot see how they could be derived by any of the standard methods† (e.g. generating functions).

† See Kendall (1949) for references.

15. *Linear coefficients*

As our last example, we take

$$\lambda_n = n\lambda \quad (n \geqslant 0), \quad \mu_n = n\mu \quad (n \geqslant 0); \qquad \lambda > 0, \quad \mu > 0, \tag{4·22}$$

so that $\lambda_0 = 0$. In this case also, $f_{ij}(t) = g_{ij}(t)$ because $\sum_1^\infty \lambda_n^{-1} = \infty$. This (and the more general example $\lambda_n = n\lambda + \kappa$, $\mu_n = n\mu$) is one of the few cases for which the explicit solutions is known. We quote the result from Kendall (1949). Define ξ_t, η_t by

$$\left.\begin{aligned} \frac{\xi_t}{\mu} = \frac{\eta_t}{\lambda} &= \frac{e^{(\lambda-\mu)t}-1}{\lambda\, e^{(\lambda-\mu)t}-\mu} \quad (\lambda \neq \mu), \\ \xi_t = \eta_t &= \frac{\lambda t}{1+\lambda t} \quad (\lambda = \mu). \end{aligned}\right\} \tag{4·23}$$

Then $f_{ij}(t)$ (for $i \geqslant 1$) is the coefficient of z^j in the expansion (in ascending powers of z) of the generating function

$$\left[\xi_t + (1-\xi_t)(1-\eta_t)\frac{z}{1-\eta_t z}\right]^i.$$

In particular,

$$\left.\begin{aligned} f_{11}(t) &= (1-\xi_t)(1-\eta_t) \\ &= (\mu-\lambda)^2 \frac{e^{(\lambda-\mu)t}}{(\lambda\, e^{(\lambda-\mu)t}-\mu)^2} \quad (\lambda \neq \mu), \\ &= (1+\lambda t)^{-2} \quad (\lambda = \mu). \end{aligned}\right\} \tag{4·24}$$

While for $\lambda \neq \mu$ we are unable to identify the polynomials $\bar{\phi}_s(x)$ involved in the spectral resolution, we can find the spectral function $\bar{\rho}(x)$ and show that the spectrum is discrete. For $\lambda = \mu$ we can determine the spectral resolution explicitly; the spectrum is continuous, extending over the whole range $-\infty < x \leqslant 0$, and the polynomials involved are the Laguerre polynomials. The spectral function $\bar{\rho}(x)$ can, in both cases, be identified by comparing the expression (4·24) for $f_{11}(t)$ with the expression (2·40*a*), i.e.

$$f_{11}(t) = -\mu^{-1} \int_{-\infty}^{0} x\, e^{tx}\, d\bar{\rho}(x). \tag{4·25}$$

If $\lambda \neq \mu$, we obtain from (4·24) that

$$f_{11}(t) = \begin{cases} \left(1-\dfrac{\mu}{\lambda}\right)^2 \displaystyle\sum_{r=1}^{\infty} r \left(\dfrac{\mu}{\lambda}\right)^{r-1} e^{-r(\lambda-\mu)t} & (\lambda > \mu), \\ \left(1-\dfrac{\lambda}{\mu}\right)^2 \displaystyle\sum_{r=1}^{\infty} r \left(\dfrac{\lambda}{\mu}\right)^{r-1} e^{-r(\mu-\lambda)t} & (\lambda < \mu). \end{cases} \tag{4·26}$$

Comparison with (4·25) shows that $\bar{\rho}(x)$ is a step-function with discontinuities d_r at

$$x = \alpha_r = -r\,|\lambda-\mu| \quad (r = 1, 2, \ldots),$$

where

$$d_r = \begin{cases} |\lambda-\mu| \dfrac{\mu^r}{\lambda^{r+1}} & (\lambda > \mu), \\ |\lambda-\mu| \dfrac{\lambda^{r-1}}{\mu^r} & (\lambda < \mu). \end{cases}$$

The spectrum is therefore discrete. It is of interest, in view of theorem 5, to note that

$$\sum_1^\infty |\alpha_r|^{-1} = \infty, \quad \sum_1^\infty |\alpha_r|^{-2} < \infty.$$

When $\lambda = \mu$ we could obtain $\bar{\rho}(x)$ by a (formal) limiting argument, letting $\lambda \to \mu$, and we should expect the spectrum to be continuous. We can check this more directly by comparing the two expressions for $f_{11}(t)$:

$$-\lambda^{-1}\int_{-\infty}^{0} x\,\mathrm{e}^{tx}\,\mathrm{d}\bar{\rho}(x) = (1+\lambda t)^{-2}.$$

Now observe that

$$(1+\lambda t)^{-2} = \int_0^\infty y\,\mathrm{e}^{-(1+\lambda t)y}\,\mathrm{d}y = \lambda^{-2}\int_0^\infty z\,\mathrm{e}^{-z/\lambda}\,\mathrm{e}^{-tz}\,\mathrm{d}z$$

$$= -\lambda^{-2}\int_{-\infty}^{0} x\,\mathrm{e}^{x/\lambda}\,\mathrm{e}^{tx}\,\mathrm{d}x.$$

Hence

$$\mathrm{d}\bar{\rho}(x) = \lambda^{-1}\,\mathrm{e}^{x/\lambda}\,\mathrm{d}x = \mathrm{d}(\mathrm{e}^{x/\lambda}),$$

and

$$\left.\begin{aligned}\bar{\rho}(x) &= \mathrm{e}^{x/\lambda}-1 \quad (x\leqslant 0),\\ &= 0 \qquad\qquad (x\geqslant 0).\end{aligned}\right\} \qquad (4{\cdot}27)$$

Since the polynomials $\bar{\phi}_s(x)$ are orthogonal with respect to the weight function

$$x\,\mathrm{d}\bar{\rho}(x) = \frac{x}{\lambda}\mathrm{e}^{x/\lambda}\,\mathrm{d}x$$

(cf. (2·40a), with $t = 0$), we should expect them to be related to the Laguerre polynomials. That this in fact is so can be seen by examining the recurrence relations

$$\bar{\phi}_0(x) = 1, \quad \bar{\phi}_1(x) = x+2\lambda,$$

$$\bar{\phi}_n(x)-(x+2n\lambda)\,\bar{\phi}_{n-1}(x)+\lambda^2 n(n-1)\,\bar{\phi}_{n-2}(x) = 0,$$

and comparing them with those for the Laguerre polynomial (Magnus & Oberhettinger 1943, p. 84) $L_n^{(\alpha)}(y)$:

$$L_0^{(\alpha)}(y) = 1, \quad L_1^{(\alpha)}(y) = \alpha+1-y,$$

$$nL_n^{(\alpha)}(y)+(y-2n-\alpha+1)\,L_{n-1}^{(\alpha)}(y)+(n+\alpha-1)\,L_{n-2}^{(\alpha)}(y) = 0.$$

It can then be seen quite easily that

$$\left.\begin{aligned}\bar{\phi}_n(x) &= \lambda^n n!\,L_n^{(1)}\!\left(-\frac{x}{\lambda}\right)\\ &= \lambda^{n-1}x^{-1}\,\mathrm{e}^{-x/\lambda}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{x/\lambda}\,x^{n+1}).\end{aligned}\right\} \qquad (4{\cdot}28)$$

In conclusion, we should like to express our indebtedness to Professor M. S. Bartlett, who not only first stimulated our interest in this problem but has always been ready to give us his valuable advice and suggestions throughout the progress of this work.

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